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NEW OSCILLATION CRITERIA FOR THIRD-ORDER NON-LINEAR NEUTRAL ADVANCED DIFFERENTIAL EQUATIONS

Dr. R. Rama[1], K. Sumaiya Parveen[2], Dr. S. Geetha[3]

[,1]Associate Professor & Head, PG & Research Department of Mathematics,
Quaid-E- Millath Government College for Women, Affiliated to University of Madras,
Chennai – 600002, Tamil Nadu, India.
[1]Corresponding author: renurama68@gmail.com
[2]Research Scholar, PG & Research Department of Mathematics,
Quaid-E- Millath Government College for Women, Affiliated to University of Madras,
Chennai – 600002, Tamil Nadu, India.
Mail id :sumaiyabarvin@gmail.com
[3]Assistant Professor, Department of Mathematics (H&S)
S.A. Engineering College,
Chennai – 600077, Tamil Nadu, India.
Mail id :srigeethamano@gmail.com

Abstract:

The purpose of this paper is to investigate the oscillatory behaviour of solutions to a class of third-order non-linear neutral advanced differential equations of the form:

$$\left(g(T)U''(T)\right)' - q(T)f(v(\tau(T))) = 0, T \geq T_0 > 0.$$

where the neutral term is defined as:

$$U(T) = v(T) + R(T)v(h(T)),$$

In contrast to the delay case, the system features an advanced argument satisfying $\tau(T) \geq T$. Assuming the canonical integral divergence condition $\int_{T_0}^{\infty} g^{-1}(s)ds = \infty$ we establish new criteria guaranteeing that all non-trivial solutions either oscillate or approach zero asymptotically. This is achieved via Riccati-type transformations and integral inequality techniques. The nonlinearity function f satisfies $f(x) \geq 0$, $x \geq k > 0$, while the operator commutativity $h \circ \tau = \tau \circ h$ facilitates a more refined estimation of R(t). The findings complement and extend current literature on third-order advanced differential equations.

Keywords: Oscillation Criteria, Third-order Differential Equation, Nonlinear advance, Neutral equations, Riccati transformation.

1.Introduction

In Mathematics, advanced differential equations are the differential equations where the derivative of the unknown function, at a certain time, is given in terms of the values of the function, at future times.

Advanced differential equations are used in many applied problems where the evolution rate depends not only on the present, but also on the future. Advanced differential equations represent the prospective memory of the future, accounting for the influence on the system of potential future actions, which are available, at the present time. Examples of typical fields where such phenomena are believed to occur include population dynamics, economics, mechanical control engineering and so on.

In this paper, we consider the third order non-linear delay differential equation of the form

$$\left(g(T)U''(T)\right)' - q(T)f\left(v(\tau(T))\right) = 0, \qquad T \geq T_0 > 0 \tag{1.1}$$

where the neutral term is defined as: $U(T) = v(T) + R(T)v(h(T)),$ (1.2)



and the following hypotheses are satisfied:
(H_1) g, q, R ∈ C([T₀, ∞),R), τ ∈ C¹([T₀, ∞), R), h ∈ C([T₀, ∞), (0, ∞)), and q(T) > 0;
(H_2) $g > 0, \int_{T_0}^{\infty} g^{-1}(s)ds = \infty$ and $0 \le R(T) \le R_0 < \infty$; (1.3)
$(H_3)\tau(T) \ge T, h(T) \ge T, h'(T) \ge h_0 > 0, \lim_{T\to\infty} \tau(T) = \lim_{T\to\infty} h(T) = \infty$, and $h \circ \tau = \tau \circ h$ (1.4)
A solution of (1.1) is defined as nontrivial function $v \in C([T_v, \infty), [0, \infty))$ with $T_v \ge T_0$, which satisfies (1.1) and $gU'' \in C([T_v, \infty), (0, \infty))$ on $[T_v, \infty)$. Those solutions v of (1.1) exist on some half-line $[T_v, \infty)$ and satisfy sup $\{|v(T)| : T_a \le T < \infty\} > 0$ for all $T_a \ge T_v$. A solution v is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called a non-oscillatory solution. An equation is oscillatory if all its solutions oscillate.

2. Preliminary results:

The following lemmas play a fundamental role in proving the subsequent results.

Lemma 2.1. [3, Lemma3] Suppose that $w_1, w_2, w_3 \in C([T_0, \infty), R)$, $\lim_{T\to\infty} w_1(T)$ exists,

$$w_3(T) \ge T,$$
$$\lim_{T\to\infty} w_3(T) = \infty \text{ for all T} \in [T_0, \infty),$$

or

$$\lim_{T\to\infty} \inf w_2(T) > -1.$$

Moreover, assume that there is a function $r \in C([T_*, \infty), R^+)$, where $T_* = min_{T\in[T_0,\infty)}\{w_3(T)\}$, such that $r(T) - w_1(T) - w_2(T)r\big(w_3(T)\big) = 0$, for all T ∈ $[T_0, \infty)$,
If $\lim_{T\to\infty} sup\, r(T) > 0$, then $\lim_{T\to\infty} w_1(T) > 0$. □

Lemma 2.2. [7] Assume that v is a non-oscillatory solution of (1.1). Then the corresponding function U satisfies

$$\text{U(T)} > 0, U''(T) > 0, \ \big(g(T)(U''(T))\big)' \le 0.$$

Moreover, there exist only two possible cases for the derivative of U:
That is

$$U'(T) < 0 \quad (2.1)$$

or

$$U'(T) > 0 \quad (2.2)$$

Theorem 2.1: [11] Suppose τ(T) is non- decreasing function such that τ(T) ≥ T for $T > T_0$ and $f \in C(R, R)$ and rf(r) > 0 $for\ r \ne 0$. Suppose that f satisfies the following condition
$\lim_{|r|\to 0} sup \frac{r}{f(r)} = Z, 0 \le Z < \infty$ (2.3)
If $\lim_{T\to\infty} inf \int_{\tau(T)}^{T} g(s)ds > \frac{Z}{e}$ hold, (2.4)
then all solutions of (1.1) oscillate. □
Suppose that Z > 0. Then, by virtue of (2.3) we can choose $T_2 \ge T_1$ such that
f(r(T)) ≥ $\frac{1}{2Z}\, r(T)$ for all $T \ge T_2$. (2.5)
In 2025, Asma AI-Jaser [6] considered the following equation
$\big(g(T)U''(T)\big)' + q(T)(v\big(\tau(T)\big)) = 0, T \ge T_0$. (2.6)



we say that (2.6) has property F if any solution v of (2.6) is either oscillatory or satisfies $v(T) \to 0$ as $T \to \infty$ (2.7)

3.MAIN RESULTS

In this section, we will adopt the following notations:

$$\gamma_1(T, T_1) = \int_{T_1}^{T} g^{-1}(S)\, dS,$$

$$\gamma_2(T, T_1) = \int_{T_1}^{T} \int_{T_1}^{S} g^{-1}(x) dx dS,$$

and $$\tilde{q}\,(\mathrm{T}) = \max\{q(\mathrm{T}), q(h(T))\}, T > T_1. \quad (3.1)$$

Lemma 3.1. Suppose that $v(T) > 0$ is a solution of equation (1.1) and

$$\lim_{T\to\infty} v(T) \neq 0. \quad (3.2)$$

If $\tau(\mathrm{T}) \geq T$ is an advanced deviating argument function such that

$$\int_{T_1}^{\infty} \int_{y}^{\infty} (g^{-1}(h(x)) \int_{x}^{\infty} \tilde{q}(\mathrm{s})\mathrm{ds})\, \mathrm{dxdy} > 2Z \quad (3.3)$$

then (2.2) holds. □

Proof: Let $v(T) > 0$ be a solution of (1.1). From (1.1), we see that

$$(\mathrm{g(T)(U''(T))})' = \mathrm{q(T)f}\left(\mathrm{v}(\tau(\mathrm{T}))\right) \geq 0.$$

Using $\mathrm{h(T)} \geq \mathrm{T}$, we have $\mathrm{g(T)}\left(\mathrm{U''(T)}\right) \leq \mathrm{g}\left(\mathrm{h(T)}\right)(\mathrm{U''(h(T))})$. (3.4)

Hence, $(\mathrm{g(T)(U''(T))})' \geq 0$ has one sign. Also $\mathrm{U''(T)}$ has one sign, and so either $\mathrm{U''(T)} > 0$ or $\mathrm{U''(T)} < 0$ for $\mathrm{T} \geq \mathrm{T_1}$. Let $\mathrm{U''(T)} < 0$. Then there is a constant $\mathrm{Z} > 0$ such that

$$\mathrm{g(T)}\left(\mathrm{U''(T)}\right) + \mathrm{Z} \leq 0. \quad (3.5)$$

Integrating (3.5) from T_1 to T , we get

$$U'(T) - U'(T_1) \leq -Z \int_{T_1}^{T} g^{-1}(s) ds.$$

Therefore, $\lim_{T\to\infty} U'(T) = -\infty$. Using the fact that $U'(T)$ and $U''(T)$ are negative, we note that $\lim_{T\to\infty} U(T) = -\infty$. This contradiction leads to $U''(T) > 0$.

Now, let $U'(T) < 0$. According to (1.1), we obtain

$$\left(g(T)U''(T)\right)' - q(T)f\left(v(\tau(T))\right) = 0$$

$$\left(g(T)U''(T)\right)' = q(T)f\left(v(\tau(T))\right)$$

Using $\mathrm{h(T)} \geq \mathrm{T}$, we have $\left(g(h(T))U''(h(T))\right)' = q(h(T))f\left(v\left(\tau(h(T))\right)\right)$

Multiply the evaluated expressions at h(T) by $\frac{R_0}{h_0}$ and add it to the expression at T



$$\left(g(T)U''(T)\right)' + \frac{R_0}{h_0}\left(g\left(h(T)\right)U''(h(T))\right)' = q(T)f\left(v\left(\tau(T)\right)\right) + \frac{R_0}{h_0}q\left(h(T)\right)f\left(v\left(\tau\left(h(T)\right)\right)\right)$$

Using (2.5) we get

$$(g(T)(U''(T)))' + \frac{R_0(g\left(h(T)\right)(U''(h(T))))'}{h_0} \leq \frac{1}{2Z}\left[q(T)v\left(\tau(T)\right) + \frac{R_0}{h_0}q\left(h(T)\right)v\left(\tau\left(h(T)\right)\right)\right].$$

Set $\widetilde{U}(T) = (g(T)(U''(T)))' + \frac{R_0(g\left(h(T)\right)(U''(h(T))))'}{h_0}$.

Then the above inequality becomes

$$\widetilde{U}(T) \leq \frac{1}{2Z}\left[\left(q(T)v\left(\tau(T)\right) + \frac{R_0}{h_0}q\left(h(T)\right)v\left(\tau\left(h(T)\right)\right)\right]\right.$$

$$\leq \frac{1}{2Z}\tilde{q}(\mathrm{T})\left[\left(v\left(\tau(T)\right) + \frac{R_0}{h_0}v\left(\tau\left(h(T)\right)\right)\right]\right.$$

In view of (1.1), we have $\widetilde{U}(T) - \frac{1}{2Z}\left[U\left(\tau(T)\right)\tilde{q}(\mathrm{T})\right] \leq 0.$ (3.6)

Integrating (3.6) from T to ∞, we get

$$g(T)(U''(T)) + \frac{R_0 g\left(h(T)\right)\left(U''\left(h(T)\right)\right)}{h_0} - \frac{1}{2Z}\int_T^\infty U\left(\tau(s)\right)\tilde{q}(\mathrm{s})\,ds \leq 0,$$

Using (3.4), we have

$$g(h(T))(U''(h(T))) - \frac{h_0}{h_0+R_0}\frac{1}{2Z}\int_T^\infty U\left(\tau(s)\right)\tilde{q}(\mathrm{s})\,ds \leq 0.$$

Based on (3.2) and Lemma 2.1, and since $\mathrm{U}(\tau(\mathrm{T})) \geq L$ and $\lim_{T\to\infty} U(T) = L > 0$, then we obtain

$$U''(h(T)) - \frac{Lh_0}{h_0+R_0}\frac{1}{2Z}g^{-1}(h(x))\int_T^\infty \tilde{q}(\mathrm{s})ds \geq 0.$$

Integrating the above inequality from T to ∞, we get

$$\frac{-U'(h(T))}{h_0} - \frac{Lh_0}{h_0+R_0}\frac{1}{2Z}\int_T^\infty g^{-1}(h(x))\int_T^\infty \tilde{q}(\mathrm{s})dsdx \geq 0. \quad (3.7)$$

Integrating again (3.7) from T_1 *to* ∞, we find

$$\frac{-U(h(T_1))}{h_0} - \frac{Lh_0}{h_0+R_0}\frac{1}{2Z}\int_{T_1}^\infty\int_y^\infty g^{-1}(h(x))\left(\int_x^\infty \tilde{q}(\mathrm{s})ds\right)dxdy \geq 0.$$

$$\int_{T_1}^\infty\int_y^\infty g^{-1}(h(x))\left(\int_x^\infty \tilde{q}(\mathrm{s})ds\right)dxdy \leq 2Z.$$



Which is a contradiction to (3.3). This completes the proof. □

Lemma 3.2. Suppose that U is satisfies (2.2) for T ≥ T_1. Then

$$g(T)\gamma_1(T,T_1) \leq \frac{U'(T)}{U''(T)} \tag{3.8}$$

and

$$g(T)\gamma_2(T,T_1) \leq \frac{U(T)}{U''(T)}\,. \tag{3.9}$$

Proof: Since $g(T)\big(U''(T)\big)$is non decreasing, we have

$$U'(T) \geq \int_T^{T_1} \frac{1}{g(S)}\Big(\, g(S)\big(U''(S)\big)\Big)\, dS \geq g(T)\big(U''(T)\big)\int_{T_1}^{T} \frac{1}{g(S)}\,\mathrm{dS}$$

$$= g(T)U''(T)\gamma_1(T,T_1).$$

This completes the proof. □

In the following results, we will assume that there exists a function $p \in C^1\big([T_0,\infty),(0,\infty)\big)$, for all $T_1 \geq T_0$.

Theorem 3.1. Suppose that (3.3) is satisfied, $\tau(\mathrm{T}) \geq h(T)$, and $\tau'(T) > 0$. If

$$\frac{1}{2Z}\lim_{T\to\infty} sup \int_{T_2}^{T}[p(s)\,\tilde{q}(\mathrm{s}) - \frac{(h_0+R_0)}{4h_0\tau'(S)p(S)\gamma_1(\tau(S),T_1)}((p'(S))_+)^2]\,\mathrm{dS} > 1, \tag{3.10}$$

for $T > T_2$, then (1.1) has property F.

Proof:

We assume that $v(T)$ to be an eventually positive solution of (1.1). Proceeding as in the proof of Lemma 3.1, from (2.2) in Lemma 2.2 and (3.6), and by using Lemma 3.2, we get (3.8). Set

$$\mu(T) = \frac{g(T)U''(T)}{U(\tau(T))}p(T) > 0. \tag{3.11}$$

From Lemma 3.1, we see that

$$\mu'(T) = \frac{g(T)U''(T)}{U(\tau(T))}p'(T) + \left(\frac{g(T)(U''(T))}{U(\tau(T))}\right)' p(T)$$

$$= \frac{g(T)U''(T)}{U(\tau(T))}p'(T) + \frac{\big(g(T)(U''(T)\big)'}{U(\tau(T))}p(T)$$

$$- \frac{g(T)(U''(T))U'(\tau(T))}{U^2(\tau(T))}\,p(T)\tau'(T) \tag{3.12}$$

Now from 3.8, $\tau(T) \geq T$ and the monotonicity of $g(T)U''(T)$, we get

$$U'\big(\tau(T)\big) \geq \big(g(\tau(T))U''\big(\tau(T)\big)\gamma_1(\tau(T),T_1)$$

$$\geq \big(g(T)U''(T)\big)\gamma_1(\tau(T),T_1). \tag{3.12 A}$$

By using (3.11) and (3.12A) in (3.12), we obtain

$$\mu'(T) \le p(T)\frac{\left(g(T)(U''(T)\right)'}{U(\tau(T))} + \frac{1}{p(T)}p'(T)\mu(T) - \frac{\gamma_1(\tau(T),T_1)}{p(T)}\tau'(T)\mu^2(T). \quad (3.13)$$

We define a new function $\Psi(T)$ as follows:

$$\Psi(T) = \frac{g(h(T))U''(h(T))}{U(\tau(T))}\,p(T). \quad (3.14)$$

That is, according to Lemma 3.1, $\Psi(T) > 0$, and

$$\Psi'(T) = p'(T)\frac{g(h(T))\left(U''(h(T))\right)}{U(\tau(T))} + p(T)\left(\frac{g(h(T))\left(U''(h(T))\right)}{U(\tau(T))}\right)'$$

$$= p'(T)\frac{g(h(T))\left(U''(h(T))\right)}{U(\tau(T))} + \frac{\left(g(h(T))(U''(h(T))\right)'}{U(\tau(T))}p(T)$$

$$- \frac{g(h(T))(U''(h((T)))U'(\tau(T))p(T)\tau'(T)}{U^2(\tau(T))}$$

$$= p'(T)g\left(h(T)\right)\frac{\left(U''(h(T))\right)}{U(\tau(T))} + \frac{\left(g(h(T))(U''\left(h((T))\right)\right)'}{U(\tau(T))}p(T)$$

$$- p(T)g\left(h(T)\right)\left(\frac{(U''(h((T)))U'(\tau(T))}{U^2(\tau(T))}\right)\tau'(T). \quad (3.15)$$

From (3.8), $\tau(T) \ge h(T)$ and the monotonicity of $g(T)U''(T)$, we get

$$U'\left(\tau(T)\right) \ge \left(g\left(\tau(T)\right)U''(\tau(T))\right)\gamma_1(\tau(T),T_1) \ge \left(g(h(T))U''(h(T))\right)\gamma_1(\tau(T),T_1), \quad (3.15\text{ A})$$

By using (3.14) and (3.15A) in (3.15), it follows that

$$\Psi'(T) \le \frac{(g(h(T))(U''(h(T)))'}{U(\tau(T))}\,p(T) + \Psi(T)\frac{\left(p'(T)\right)_+}{p(T)} - \tau'(T)\Psi^2(T)\frac{\gamma_1(\tau(T),T_1)}{p(T)}. \quad (3.16)$$

Combining (3.13) and (3.16), we obtain

$$\mu'(T) + \Psi'(T)\frac{R_0(T)}{h_0} \le \frac{p(T)}{U(\tau(T))}\left[\left(g(T)\left(U''(T)\right)\right)' + \frac{R_0(g(h(T))(U''\left(h(T)\right))'}{h_0}\right]$$

$$+\mu(T)\frac{p'(T)}{p(T)} - \mu^2(T)\frac{\gamma_1(\tau(T),T_1)}{p(T)}\tau'(T)$$

$$+\frac{R_0}{h_0}\left[\Psi(T)\frac{p'(T)}{p(T)} - \Psi^2(T)\frac{\gamma_1(\tau(T),T_1)}{p(T)}\tau'(T)\right] \quad (3.17)$$

It follows from (3.6) and (3.17), we have

$$\mu'(T) + \Psi'(T)\frac{R_0(T)}{h_0} \le \frac{-p(T)\tilde{q}(T)}{2Z} + \mu(T)\frac{\left(p'(T)\right)_+}{p(T)} - \mu^2(T)\frac{\gamma_1(\tau(T),T_1)}{p(T)}\tau'(T)$$

$$+\frac{R_0}{h_0}\left[\Psi(T)\frac{p'(T)}{p(T)} - \Psi^2(T)\frac{\gamma_1(\tau(T),T_1)}{p(T)}\tau'(T)\right] \quad (3.18)$$

By applying the inequality stated in [15], namely,



$$\frac{D^2}{4C} \geq Dx - Cx^2, C > 0. \tag{3.19}$$

By taking $C = \frac{\gamma_1(\tau(T),T_1)\tau'(T)}{p(T)}$, $x = \mu(T)$, $D = \frac{p'(T)}{p(T)}$, we have

$$\frac{p(T)}{4\gamma_1(\tau(T),T_1)\tau'(T)}\left(\frac{(p'(T))_+}{p(T)}\right)^2 \geq \mu(T)\left(\frac{(p'(T))_+}{p(T)}\right) - \mu^2(T)\left(\frac{\gamma_1(\tau(T),T_1)\tau'(T)}{p(T)}\right)$$

and

$$\frac{p(T)}{4\gamma_1(\tau(T),T_1)\tau'(T)}\left(\frac{(p'(T))_+}{p(T)}\right)^2 \geq \Psi(T)\left(\frac{(p'(T))_+}{p(T)}\right) - \Psi^2(T)\left(\frac{\gamma_1(\tau(T),T_1)\tau'(T)}{p(T)}\right),$$

which implies that

$$\mu'(T) + \Psi'(T)\frac{R_0(T)}{h_0} \leq \frac{-p(T)\tilde{q}(T)}{2Z} + \frac{1}{4}\frac{((p'(T))_+)^2}{p(T)\tau'(T)\gamma_1(\tau(T),T_1)} + \frac{R_0}{4h_0}\frac{((p'(T))_+)^2}{p(T)\tau'(T)\gamma_1(\tau(T),T_1)} \tag{3.20}$$

Integrating from T_2 to T, we see that

$$\frac{1}{2Z}\int_{T_2}^{T}\left(p(s)\tilde{q}(s) - \frac{h_0+R_0}{4h_0}\frac{((p'(T))_+)^2}{p(T)\tau'(s)\gamma_1(\tau(s),T_1)}\right)\mathrm{ds} \leq \left(\mu(T_2) + \frac{R_0\Psi(T_2)}{h_0}\right)$$

$$\int_{T_2}^{T}\left(p(s)\tilde{q}(s) - \frac{h_0+R_0}{4h_0}\frac{((p'(T))_+)^2}{p(T)\tau'(s)\gamma_1(\tau(s),T_1)}\right)\mathrm{ds} \leq 2Z\left(\mu(T_2) + \frac{R_0\Psi(T_2)}{h_0}\right)$$

Taking the limit superior as $T \to \infty$, we get

$$\lim_{T\to\infty} sup \int_{T_2}^{T}[p(S)\,\tilde{q}(S) - \frac{(h_0+R_0)}{4h_0\tau'(S)p(S)\gamma_1(\tau(S),T_1)}((p'(S))_+)^2]\mathrm{dS} \leq 2Z,$$

which is contradiction with (3.10). The proof is complete. □

Theorem 3.2:

Let (3.3) be satisfied, b(t) ≥ $l(t)$. If

$$\frac{1}{2Z}\lim_{T\to\infty} sup \int_{T_2}^{T}[\,p(S)\,\tilde{q}(S) - \frac{(h_0+R_0)((p'(S))_+)^2}{4h_0h'(S)p(S)\gamma_1(h(S),T_1)}]\mathrm{dS} > 1, \tag{3.21}$$

for $T > T_2$, then (1.1) has property F.

Proof:

We assume that v(T) > 0 to be an eventually positive solution of (1.1). Proceeding as in the proof of Lemma 3.1, from (2.2) in Lemma 2.2 and (3.6), and by using Lemma 3.2, we get (3.8). Set

$$\mu(T) = \frac{g(T)U''(T)}{U(h(T))}p(T) > 0. \tag{3.22}$$

From Lemma 3.1, we see that

$$\mu'(T) = \frac{g(T)U''(T)}{U(h(T))}p'(T) + \left(\frac{g(T)(U''(T))}{U(h(T))}\right)'p(T)$$

$$= \frac{g(T)U''(T)}{U(h(T))}p'(T)+\frac{\left(g(T)(U''(T)\right)'}{U(h(T))}p(T)$$
$$-\frac{g(T)(U''(T))U'(h(T))}{U^2(h(T))}\,p(T)h'(T) \qquad (3.23)$$

which in view of (2.2), (3.8), and $h(T) \geq T$, we get

$$U'\big(h(T)\big) \geq \big(g(h(T))U''\big(h(T)\big)\gamma_1(h(T),T_1)$$
$$\geq \big(g(T)U''(T)\big)\gamma_1(h(T),T_1). \qquad (3.23\ A)$$

By using (3.22) and (3.23 A) in (3.23), we obtain

$$\mu'(T) \leq p(T)\frac{\left(g(T)(U''(T)\right)'}{U(h(T))}+\frac{1}{p(T)}p'(T)\mu(T)-\frac{\gamma_1(h(T),T_1)}{p(T)}h'(T)\mu^2(T). \qquad (3.24)$$

We define a new function $\Psi(T)$ as follows:

$$\Psi(T) = \frac{g\big(h(T)\big)U''\big(h(T)\big)}{U(h(T))}\,p(T). \qquad (3.25)$$

That is, according to Lemma 3.1, $\Psi(T) > 0$, and

$$\Psi'(T) = p'(T)\frac{g\big(h(T)\big)\big(U''\big(h(T)\big)\big)}{U\big(h(T)\big)}+p(T)\left(\frac{g\big(h(T)\big)\big(U''\big(h(T)\big)\big)}{U\big(h(T)\big)}\right)'$$
$$= p'(T)\frac{g\big(h(T)\big)\big(U''\big(h(T)\big)\big)}{U\big(h(T)\big)}+\frac{\left(g\big(h(T)\big)(U''\big(h(T)\big)\big)\right)'}{U\big(h(T)\big)}p(T)$$
$$-\frac{g(h(T))(U''(h((T)))U'(h(T))p(T)h'(T)}{U^2(h(T))}$$
$$= p'(T)g\big(h(T)\big)\frac{\left(U''(h(T))\right)}{U(h(T))}+\frac{\left(g(h(T))(U''\big(h((T))\big)\right)'}{U(h(T))}p(T)$$
$$-p(T)g\big(h(T)\big)\left(\frac{(U''(h((T)))U'(h(T))}{U^2(h(T))}\right)h'(T). \qquad (3.26)$$

Using (2.2) and (3.8) , we have

$$U'\big(h(T)\big) \geq (g(h(T))U''(h(T)))\gamma_1(h(T),T_1),$$

From (3.25) and (3.26), it follows that

$$\Psi'(T) \leq \frac{(g\big(h(T)\big)(U''(h(T)))'}{U\big(h(T)\big)}\,p(T)+\Psi(T)\frac{\left(p'(T)\right)_+}{p(T)}-\,h'(T)\Psi^2(T)\frac{\gamma_1(h(T),T_1)}{p(T)}. \qquad (3.27)$$

Combing (3.24) and (3.27), we obtain

$$\mu'(t)+\Psi'(T)\frac{R_0(T)}{h_0} \leq \frac{p(T)}{U\big(h(T)\big)}\left[\Big(g(T)\big(U''(T)\big)\Big)'+\frac{R_0(g(h(T))(U''\big(h(T)\big)))'}{h_0}\right]$$



$$+\mu(T)\frac{(p'(T))}{p(T)} - \mu^2(T)\frac{\gamma_1(h(T),T_1)}{p(T)}h'(T)$$

$$+\frac{R_0}{h_0}\left[\Psi(T)\frac{\left(p'(T)\right)}{p(T)} - \Psi^2(T)\frac{\gamma_1(h(T),T_1)}{p(T)}h'(T)\right] \quad (3.28)$$

From (2.2), (3.6) and (3.28), and $\tau(T) \geq h(T)$, we obtain

$$\mu'(T) + \Psi'(T)\frac{R_0(T)}{h_0} \leq \frac{-p(T)\tilde{q}(T)}{2Z} + \mu(T)\frac{\left(p'(T)\right)_+}{p(T)} - \mu^2(T)\frac{\gamma_1(\tau(T),T_1)}{p(T)}\tau'(T)$$

$$+\frac{R_0}{h_0}\left[\Psi(T)\frac{p'(T)}{p(T)} - \Psi^2(T)\frac{\gamma_1(h(T),T_1)}{p(T)}h'(T)\right] \quad (3.29)$$

By (3.29) and (3.19), we get

$$\mu'(T) + \Psi'(T)\frac{R_0(T)}{h_0} \leq \frac{-p(T)\tilde{q}(T)}{2Z} + \frac{1}{4}\frac{\left((p'(T))_+\right)^2}{p(T)h'(T)\gamma_1(h(T),T_1)} + \frac{R_0}{4h_0}\frac{\left((p'(T))_+\right)^2}{p(T)h'(T)\gamma_1(h(T),T_1)} \quad (3.30)$$

Integrating (3.30) from T_2 to T, we see that

$$\frac{1}{2Z}\int_{T_2}^{T}\left(p(S)\tilde{q}(S) - \frac{h_0+R_0}{4h_0}\frac{\left((p'(T))_+\right)^2}{p(T)h'(S)\gamma_1(h(S),T_1)}\right)\mathrm{ds} \leq \left(\mu(T_2) + \frac{R_0\Psi(T_2)}{h_0}\right)$$

$$\int_{T_2}^{T}\left(p(S)\tilde{q}(S) - \frac{h_0+R_0}{4h_0}\frac{\left((p'(T))_+\right)^2}{p(T)h'(S)\gamma_1(h(S),T_1)}\right)\mathrm{ds} \leq 2Z\left(\mu(T_2) + \frac{R_0\Psi(T_2)}{h_0}\right)$$

Taking the limit superior as $T \to \infty$, we get

$$\frac{1}{2Z}\lim_{T\to\infty} sup\int_{T_2}^{T}[p(S)\,\tilde{q}\,(S) - \frac{(h_0+R_0)}{4h_0h'(S)p(S)\gamma_1(h(S),T_1)}((p'(S))_+)^2]\mathrm{dS} \leq 1,$$

which is contradiction with (3.21). The proof is complete. □

Remark 3.1: From Theorems 3.1 and 3.2, we can get some oscillation criteria for (1.1) with different choices of p.

3.1. Absence of solutions in case (2.1)

Theorem 3.3:

If there exists a function $\xi(t) \in C\left(\left[T_0,\infty\right),(0,\infty)\right)$, $\tau(\mathrm{T}) > \xi(T)$, and $h^{-1}\left(\xi(T)\right) > T$ such that the first – order advanced differential equation

$$J'(T) + \frac{h_0\,\tilde{q}(T)\gamma_2\left(\xi(T),\tau(T)\right)}{h_0+R_0}J\left(h^{-1}\left(\xi(T)\right)\right) = 0 \quad (3.31)$$

is oscillatory, then case (2.2) cannot hold.

Proof:

Assume that $x > 0$ is a solution of (1.1). Let (2.2) in Lemma 2.2 hold. By (3.6), we obtain

$$\widetilde{U}\,(\mathrm{T}) - \frac{1}{2Z}\,U(\tau(T))\tilde{q}(T) \leq 0 \quad (3.32)$$



It follows from $g(T)(U''(T)) \geq 0$ that

$$-U'(x) \geq U'(y) - U'(x) = \int_y^x U''(s)\frac{g(s)}{g(s)}\,ds$$

$$\geq\ g(y)U''(y)\int_y^x \frac{1}{g(s)}ds, \text{ for } y \geq x \geq T_1.$$

From (3.9), we have

$$U(\tau(T)) \geq\ g(\xi)(U''(\xi))\gamma_2(\xi, \tau(T)),$$

Which from (3.32) implies that

$$\widetilde{U}\,(T) \leq \frac{1}{2Z}(-\tilde{q}(T)\gamma_2(\xi, \tau(T))\ g(\xi)(U''(\xi))) \tag{3.33}$$

Since $g(T)U''(T)$ is nondecreasing, we get

$$\widetilde{U}\,(T) \leq \frac{(h_0+R_0)\ g(h(T))U''(h(T))}{h_0} \tag{3.34}$$

Hence, we have

$$g(h(t))U''(h(T)) \geq \frac{h_0\widetilde{U}\left(\left(h^{-1}(\xi(T))\right)\right)}{(h_0+R_0)} \tag{3.35}$$

Combining (3.33) and (3.35), we note that w(T) = $\widetilde{U}\,(T)$ is a positive solution of the first-order advanced differential inequality

$$\left(\widetilde{w}\,(T)\right)' \leq \frac{1}{2Z}\frac{h_0\,\tilde{q}(T)\gamma_2\left(\xi(T),\tau(T)\right)}{h_0+R_0}\widetilde{w}\left(h^{-1}\left(\xi(T)\right)\right)$$

By a standard comparison theorem equation (3.31) must also possess a positive solution, which contradicts the hypothesis that (3.31) is oscillatory.

The proof is complete.

Theorem 3.4:

If there exists a function $q(T) \in C\left([t_0, \infty), (0, \infty)\right), q(T) > T$ and $\tau(\mathrm{T}) > h(q(T))$ such that

$$\frac{1}{2Z}\lim_{T\to\infty}\inf\gamma_2(h(\,q(T)), \tau(T))\int_{q(T)}^{T}\tilde{q}(s)ds > \left(\frac{h_0+R_0}{h_0}\right) \tag{3.36}$$

Then case (2.1) cannot hold.

Proof:

$$\widetilde{U}(T) = \ (g(T)(U''\,(T)))' + \frac{R_0(g\left(h(T)\right)(U''(h(T))))'}{h_0}$$

$$\widetilde{U}(T) - \frac{1}{2Z}\left[U\left(\tau(T)\right)\tilde{q}(\mathrm{T})\right]\ \leq 0.$$

$$(g(T)(U''\,(T)))' + \frac{R_0(g\left(h(T)\right)(U''(h(T))))'}{h_0} \leq \frac{1}{2Z}\left[U\left(\tau(T)\right)\tilde{q}(\mathrm{T})\right]$$

Integrating from $q(T)$ to T and using the monotonicity properties of U, we have



$$\left[g(T)(U''(T)) + \frac{R_0 g\big(h(T)\big)\Big(U''\big(h(T)\big)\Big)}{h_0}\right]_{q(T)}^{T} \leq \frac{1}{2Z}\int_{q(T)}^{T} U\big(\tau(s)\big)\tilde{q}(s)\,ds$$

$$g(T)(U''(T)) + \frac{R_0 g\big(h(T)\big)\Big(U''\big(h(T)\big)\Big)}{h_0} - \left[g(q(T))(U''(q(T))) + \frac{R_0 g\big(h(q(T))\big)\Big(U''\big(h(q(T))\big)\Big)}{h_0}\right] \leq \frac{1}{2Z}\int_{q(T)}^{T} U\big(\tau(s)\big)\tilde{q}(s)\,ds$$

$$-\left[g\big(q(T)\big)\big(U''\big(q(T)\big)\big) + \frac{R_0 g\left(h\big(q(T)\big)\right)\Big(U''\left(h\big(q(T)\big)\right)\Big)}{h_0}\right] \leq -\frac{1}{2Z}\int_{q(T)}^{T} U\big(\tau(s)\big)\tilde{q}(s)\,ds$$

Since $g(T)(U''(T)) \leq 0$ and $h(T) \geq T$ and since $U'(T) < 0$.

$$\left[g\big(q(T)\big)\big(U''\big(q(T)\big)\big) + \frac{R_0 g\left(h\big(q(T)\big)\right)\Big(U''\left(h\big(q(T)\big)\right)\Big)}{h_0}\right] \geq \frac{1}{2Z}\int_{q(T)}^{T} U\big(\tau(s)\big)\tilde{q}(s)\,ds$$

Using (3.9), we have

$$g(h(q(T)))(U''(h(q(T))))\left(\tfrac{h_0+R_0}{h_0}\right) \geq \tfrac{1}{2Z}\, g(\tau(T)r_2(\tau(T),\tau(T_1))U''(\tau(T))\int_{q(T)}^{T}\tilde{q}(s)ds. \qquad (3.37)$$

Since $h(q(T) > \tau(T)$

$$g(h(q(T)))(U''(h(q(T))))\left(\frac{h_0+R_0}{h_0}\right) \geq \frac{1}{2Z}\,\left(g(h(q(T)))(U''(h(q(T))))\gamma_2\big(h\big(q(T)\big),\tau(T)\big)\right)\int_{q(T)}^{T}\tilde{q}(s)ds.$$

that is,

$$\left(\frac{h_0+R_0}{h_0}\right) \geq \frac{\gamma_2\big(h\big(q(T)\big),\tau(T)\big)}{2Z}\int_{T}^{q(T)}\tilde{q}(s)ds.$$

Taking limit inferior on both sides as $t \to \infty$, we get

$$\frac{1}{2Z}\liminf_{T\to\infty}\gamma_2(h(\,q(T)),\tau(T))\int_{q(T)}^{T}\tilde{q}(s)ds \leq \left(\frac{h_0+R_0}{h_0}\right)$$

which is contradiction with (3.36), the proof is complete. □



3.2. Philo's-type oscillation criteria

In this section, we will establish some Philo's-type oscillation results for (1.1).

Definition 3.1. Assume that $D_0 = \{(T,S): T > S \geq T_0\}$ and $D = \{(T,S): T \geq S \geq T_0\}$, and F$\in C([D,R)$ satisfies the following:

(I) $F(T,S) > 0, (T,S) \in D_{0,}$ and $F(T,T) = 0$, for $T \geq T_0$;

(II) The partial derivative $\frac{\partial F}{\partial S}$ exists on $D_{0,}$is continuous, and there exists a continuous function $N(T,S)$ such that $\frac{\partial F}{\partial S} = -N(T,S)\sqrt{F(T,S)} \leq 0$ for (T, S)$\in D_{0,}$.

Under these assumptions, the function F satisfies the property F.

We introduce the following functions:

$$K_1(T,S) = \frac{\alpha(T,S)}{\gamma_1(\tau(T),T_1)\tau'(T)} \quad , \quad K_2(T,S) = \frac{\alpha(T,S)}{\gamma_2(\tau(T),T_1)\tau'(T)}$$

and

$$K_3(T,S) = \frac{\alpha(T,S)}{\gamma_1(h(T),T_1)h'(T)} \quad , \quad K_4(T,S) = \frac{\alpha(T,S)}{\gamma_2(h(T),T_1)h'(T)} \quad ,$$

where $\alpha(T,S) = \frac{(h_0+R_0)(N(T,S))^2}{4h_0 p(T)}$

Furthermore, in this section, suppose that F$\in C(D,R)$ has the property F and there exists a function $p \in C^1([t_0,\infty),(0,\infty))$,for all $T_1 \geq T_0$.

Theorem 3.5:

Assume that (3.3) is satisfied, $\tau'(T) > 0$, and $\tau(T) \geq h(T)$. If

$$N(T,S)\frac{(F(T,S))^{\frac{1}{2}}}{p(S)} = -\frac{\partial}{\partial S}F(T,S) - \frac{p'(S)}{p(S)}F(T,S), \qquad (T,S) \in D_{0,} \tag{3.39}$$

and

$$\frac{1}{2Z}\lim_{T\to\infty} sup F^{-1}(T,T_2)\int_{T_2}^{T}\big(F(T,S)p(S)\tilde{q}(S) - K_1(T,S)\big)ds > 1 \tag{3.40}$$

For $T > T_2$, then (1.1) has property F.

Proof: Assume that $v > 0$ be a solution of (1.1). As in Theorem 3.1, by defining μ and Ψ, we get (3.18). From (3.18) and by replacing $(p'(T))_+$ by $p'(T)$, we obtain

$$\begin{aligned}\frac{1}{2Z}p(T)\tilde{q}(T) \leq -\mu'(T) &- \frac{R_0\Psi'(T)}{h_0} + \frac{p'(T)\mu(T)}{p(T)} \\ &- \frac{\gamma_1(\tau(S),T_1)\tau'(S)}{p(T)}\mu^2(T) \\ &+ \frac{R_0}{h_0}\left(\frac{(p'(T))\Psi(T)}{p(T)} - \frac{\gamma_1(\tau(S),T_1)\tau'(S)}{p(T)}\Psi^2(T)\right).\end{aligned} \tag{3.41}$$

Multiply both sides by F (T, S), then integrate from T_2 to T. It follows that

$$\frac{1}{2Z}\int_{T_2}^{T} p(S)F(T,S)\tilde{q}(S)\,dS \le -\int_{T_2}^{T} F(T,S)\mu'(S)\,dS + \int_{T_2}^{T}\frac{F(T,S)p'(S)\mu(S)}{p(S)}dS$$
$$-\int_{T_2}^{T}\frac{F(T,S)\gamma_1(\tau(S),T_1)\tau'(S)}{p(S)}\mu^2(T)ds$$
$$-\frac{R_0}{h_0}\left(\int_{T_2}^{T} F(T,S)\Psi'(S)\,dS - \int_{T_2}^{T}\frac{F(T,S)p'(S)\Psi(S)}{p(S)}\,dS\right)$$
$$-\frac{R_0}{h_0}\int_{T}^{T_2}\frac{F(T,S)\gamma_1(\tau(S),T_1)\tau'(S)}{p(S)}\Psi^2(S)dS.$$

Thus, we have

$$\frac{1}{2Z}\int_{T_2}^{T} p(S)F(T,S)\tilde{q}(S)\,dS \le F(T,T_2)\mu(T_2)$$
$$-\int_{T_2}^{T}\left(-\frac{\partial}{\partial S}F(T,S) - \frac{p'(S)}{p(S)}F(T,S)\right)\mu(S)dS$$
$$-\int_{T_2}^{T}\frac{F(T,S)\gamma_1(\tau(S),T_1)\tau'(S)}{p(S)}\mu^2(S)dS + \frac{R_0}{h_0}\mathrm{F}\,(\mathrm{T},T_2)\Psi(T_2)$$
$$-\frac{R_0}{h_0}\int_{T_2}^{T}\left(-\frac{\partial}{\partial S}F(T,S) - \frac{1}{p(S)}p'(S)F(T,S)\right)\Psi(S)dS$$
$$-\frac{R_0}{h_0}\int_{T_2}^{T}\frac{F(T,S)\gamma_1(\tau(S),T_1)\tau'(S)}{p(S)}\Psi^2(S)dS.$$

that is

$$\frac{1}{2Z}\int_{T}^{T_2} F(T,S)p(S)\tilde{q}(S)\,dS \le F(T,T_2)\mu(T_2) + \frac{R_0}{h_0}F(T,T_2)\Psi(T_2)$$
$$+\int_{T_2}^{T}\left[\frac{N(T,S)\left(F(T,S)\right)^{\frac{1}{2}}}{p(S)}\,\mu(s) - \frac{F(T,S)\gamma_1(\tau(S),T_1)\tau'(T)}{p(S)}\mu^2(S)\right]ds$$
$$+\frac{R_0}{h_0}\int_{T_2}^{T}\left[\frac{N(T,S)\left(F(T,S)\right)^{\frac{1}{2}}}{p(S)}\,\Psi(s) - \frac{F(T,S)\gamma_1(\tau(S),T_1)\tau'(T)}{p(S)}\Psi^2(S)\right]ds. \qquad (3.42)$$

Using (3.42) and (3.19), we find

$$F^{-1}(T,T_2)\int_{T_2}^{T}\left[F(T,S)p(S)\tilde{q}(S) - \frac{h_0+R_0}{4h_0}\frac{\left(N(T,S)\right)^2}{\left(p(S)\gamma_1(\tau(S),T_1)\tau'(S)\right)}\right]\mathrm{dS} \le 2Z\left(\mu(T_2) + \frac{R_0\Psi(T_2)}{h_0}\right),$$

Taking limit superior, we get

$$\frac{1}{2Z}\lim_{T\to\infty} supF^{-1}(T,T_2)\int_{T_2}^{T}\left(F(T,S)p(S)\tilde{q}(S) - K_1(T,S)\right)dS \le 1$$

and this contradicts (3.40). The proof is complete. □



Now we give few more results without proof.

Theorem 3.6: Assume that (3.3) is satisfied, $\tau'(T) > 0$, and $\tau(T) \leq h(T)$. If

$$\mathrm{N(T,S)}\frac{(F(T,S))^{\frac{1}{2}}}{p(S)} = -\frac{\partial}{\partial S}F(T,S) - \frac{p'(S)}{p(S)}F(T,S), \qquad (T,S) \in D_{0,} \tag{3.43}$$

and

$$\frac{1}{2Z}\lim_{T\to\infty} supF^{-1}(T,T_2)\int_{T_2}^{T}\big(F(T,S)p(S)\tilde{q}(S) - K_1(T,S)\big)ds > 1 \tag{3.44}$$

For $T > T_2$, then (1.1) has property F.

Theorem 3.7:

Let (3.3) and (3.39) be satisfied, and $\tau(T) \geq h(T)$,If

$$\lim_{\mathrm{T}\to\infty} \mathrm{supF^{-1}(T,T_2)}\int_{T_2}^{T}\big(\mathrm{F(T,S)p(S)\tilde{q}(S) - K_3(T,S)}\big)\mathrm{dS} > 2Z \tag{3.45}$$

for $T > T_2$, then (1.1) has property F.

Theorem 3.8:

Let (3.3) and (3.43) be satisfied, and $\tau(T) \geq h(T)$,If

$$\lim_{\mathrm{T}\to\infty} \mathrm{supF^{-1}(T,T_2)}\int_{T_2}^{T}\big(\mathrm{F(T,S)p(S)\tilde{q}(S) - K_4(T,S)}\big)\mathrm{ds}\,S > 2\mathrm{Z} \tag{3.46}$$

for $T > T_2$,then any solution v of (1.1) is oscillatory or satisfies (2.7).

Example 4.1:

Consider the third order non-linear advanced differential equation

$$\big(T^{-2}([v(T) + R_0v(1.2T)]'')\big)' - (T^{-4}\lambda)v^4(1.2T) = 0, \lambda > 0, R_0 > 0 \tag{4.1}$$

Comparing (4.1) with (1.1) we have , $g(T) = T^{-2}$, τ(T)=1.1T, h(T) = 1.2T, $h_0 = 1.2$, and

$$q(T) = \lambda T^{-4}, \text{where} f(v) = v^4$$

Since $\tau(T) = 1.1T \geq 1.2T = h(T)$.

That is, we have $\tilde{q}(T) = \max\{q(T), q((h(T))\} = \max\{\lambda T^{-4}, \lambda(1.2T)^{-4}\} = \lambda T^{-4}$.

$$and\ \gamma_1(T,T_1) = \int_{T_1}^{T} S^2 dS \geq \frac{T^3 - {T_1}^3}{3} \geq T^3$$

Putting $p(T) = T^3$ and according to Theorem 3.1, we find

$$\lim_{T\to\infty} sup\int_{T_2}^{T}[p(S)\,\tilde{q}\,(\mathrm{S}) - \frac{h_0+R_0}{4h_0\tau'(S)p(S)\gamma_1(\tau(S),T_1)}((p'(S))_+)^2]\mathrm{dS} > 2Z$$

$$p(S) = S^3\,, p'(S) = 3S^2.$$

$$\tilde{q}\,(\mathrm{S}) = \lambda S^{-4}, \tau'(S) = 1.1, \qquad \gamma_1(\tau(S),T_1) = (1.1S)^3 = 1.3331S^3$$



$$\lim_{T\to\infty} sup \int_{T_2}^{T} \left(S^3(\lambda S^{-4}) - \frac{1.2 + R_0}{4((1.2)(1.1)(s^3)(1.3331S^3)} (3S^2)^2 \right) dS > 2Z,$$

$$\lim_{T\to\infty} sup \int_{T_2}^{T} \left(\lambda S^{-1} - \frac{9(1.2 + R_0)}{7.02768} S^{-2} \right) dS > 2Z,$$

Evaluating the definite integral from T_2 to T

$$\lim_{T\to\infty} sup \left(\lambda \ln\left(\frac{T}{T_2}\right) + \frac{9(1.2+R_0)}{7.02768} \left(\frac{1}{T_2} - \frac{1}{T}\right) \right) > 2Z.$$

As T→ ∞ the term $\frac{1}{T} \to 0$ and $\ln\left(\frac{T}{T_2}\right) \to \infty$.

ln(T) → ∞, the limit superior will automatically grow infinitely large and easily cross the finite threshold of $> 2Z$. Provided that the coefficient λ remains strictly positive.

Thus, we conclude that any solution of (4.1) is oscillatory or satisfies (2.7) if

$$\lambda > \frac{9(1.2+R_0)}{7.02768} \qquad (4.2)$$

Example 4.2:

Consider the third order non-linear advanced differential equation

$$\left(T^{-1} \left(\left[v(t) + R_0 v\left(\frac{3}{4}T\right) \right]'' \right) \right)' - (T^{-2}\lambda) v^3 \left(\frac{3}{2}T\right) = 0, T \geq T_1 > 0 \qquad (4.3)$$

where $\lambda > 0, R = R_0 > 0$. $f(v) = v^3,$ makes the equation strictly non – linear.

$g(T) = T^{-1}$, τ(T) = $\frac{3}{2}$T, h(t) = $\frac{3}{4}$T, $h_0 = h'(t) = \frac{3}{4}$, and $q(T) = \lambda T^{-2}$.

That is, we have $\tilde{q}(T) = \max\{ q(T), q((h(T))\} = \max\left\{\lambda T^{-2}, \lambda\left(\frac{3}{4}T\right)^{-2}\right\} = \lambda T^{-2}$.

$$and\ \gamma_1(T, T_1) = \int_{T_1}^{T} SdS \geq \frac{T^2 - {T_1}^2}{2} \geq \frac{T^2}{2}$$

Putting $p(T) = T^2$ and according to Theorem 3.2, we have to show that

$$\frac{1}{2Z} \lim_{T\to\infty} sup \int_{T_2}^{T} [\, p(S)\, \tilde{q}(S) - \frac{(h_0+R_0)\left((p'(S))_+\right)^2}{4h_0 h'(S) p(S) \gamma_1(h(S), T_1)}] \mathrm{dS} > 1,$$

$$p(S) = S^2, \quad , p'(S) = 2S.$$

$$\tilde{q}\,(S) = \lambda S^{-2},\ h'(S) = \frac{3}{4}, \qquad \gamma_1(h(S), T_1) = \frac{(h(s))^2}{2} = \frac{1}{2}\left(\frac{3}{4}S\right)^2 = \frac{9}{32}S^2$$

Now take



$$\frac{1}{2Z}\lim_{T\to\infty} sup \int_{T_2}^{T}[\, p(S)\,\tilde{q}(S) - \frac{(h_0+R_0)\left((p'(S))_+\right)^2}{4h_0 h'(S)p(S)\gamma_1(h(S),T_1)}]\mathrm{dS}$$ Type equation here.

$$= \lim_{T\to\infty} sup \int_{T_2}^{T}[\; s^2\;(\lambda s^{-2}) - \frac{\left(\frac{3}{4}+R_0\right)(2S)^2}{4\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)S^2\left(\frac{9}{32}S^2\right)}]\mathrm{ds}$$

$$= \lim_{T\to\infty} sup \int_{T_2}^{T}\left(\lambda - \frac{512R_0+384)}{81S^2}\right)\mathrm{dS}$$ Type equation here.

Evaluating the definite integral from T_2 to T

$$\lim_{T\to\infty} sup \int_{T_2}^{T}\left(\lambda\,(\mathrm{T}-\,T_2) + \frac{512R_0+384)}{81}\left(\frac{1}{T}-\frac{1}{T_2}\right)\right)$$

As T→ ∞ the term λT grows infinitely, while $\frac{1}{T}$ vanishes to 0.

$$\lim_{T\to\infty} sup \int_{T_2}^{T}\left(\lambda\,(\mathrm{T}-\,T_2) - \frac{512R_0+384)}{81}\right) = \infty$$

Since the limit goes to infinity, it is guaranteed that for any arbitrarily large positive constant Z.

$$\frac{1}{2Z}\lim_{T\to\infty} sup \int_{T_2}^{T}[\, p(S)\,\tilde{q}(S) - \frac{(h_0+R_0)\left((p'(S))_+\right)^2}{4h_0 h'(S)p(S)\gamma_1(h(S),T_1)}]\mathrm{dS} > 1,$$

Hence any solution of (4.3) is either oscillatory or satisfies (2.7).

5. CONCLUSION

In this paper, the oscillatory behaviour of third order neutral Advanced differential Equations are discussed. New sufficient conditions namely (3.3), (3.10), (3.21), (3.31), (3.36) establish the oscillations of the solutions of (1.1). Additionally, we explore the Philo's – type oscillatory behaviour of Equation (1.1), through theorems (3.5 – 3.8). Moreover, examples are given to substantiate theorem (3.1) and (3.2).

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