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A NEW INTELLIGENT DECISION-MAKING FRAMEWORK FOR THE DIAGNOSIS OF CHRONIC CANCER USING CUBIC BIPOLAR PYTHAGOREAN FUZZY TOPOLOGY AND MULTI-CRITERIA DECISION – MAKING

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Abstract: This paper introduces the **Cubic Bipolar Pythagorean Fuzzy Set (CBPFS)**, a novel framework for modelling uncertainty using fuzzy intervals and fuzzy numbers. The fundamental properties and operations of CBPFSs are presented, followed by the development of **Cubic Bipolar Pythagorean Fuzzy Topology (CBPF Topology)** based on P-order. Classical topological concepts, including open sets, closed sets, interior, exterior, frontier, neighborhoods, bases, and subspaces, are extended to the CBPF setting. The proposed topology provides an effective framework for handling uncertainty in complex decision-making problems. Furthermore, a **CRITIC–MAIRCA** multi-criteria group decision-making method is developed within the CBPF topological framework and applied to chronic cancer disease diagnosis. Comparative analysis demonstrates the effectiveness and reliability of the proposed approach over existing methods.

Keywords: CBPFSs, CBPF Topology with P-order, CBPF CRITIC - MAIRCA

I. INTRODUCTION

The concept of fuzzy set theory, which is offered by Zadeh [1], presents an excellent mathematical technique for dealing with uncertainty and imprecision. Later on, many other types of fuzzy sets have been formulated for giving a more flexible representation of uncertain data. Atanassov [2] presented the idea of intuitionistic fuzzy sets through the introduction of both membership degree and non-membership degree. Yager [3], [4] suggested the concept of Pythagorean fuzzy sets, which give a more general representation of uncertain data than intuitionistic fuzzy sets by considering a relaxed constraint on them. Bipolar fuzzy sets have been introduced by Zhang [5], which allow the representation of positive as well as negative data at a time.

The cubic fuzzy set has been introduced by Jun et al. [8] by incorporating the interval and fuzzy membership functions into a single framework. Cubic fuzzy sets based fuzzy approaches have been extensively studied and used in decision-making and pattern recognition processes after this study. For instance, Garg and Kaur [9] defined distance measures for cubic intuitionistic fuzzy sets and used them in pattern recognition and disease diagnosis applications. Garg and Kaur [12] explored the basic properties of cubic intuitionistic fuzzy sets.

Topology is one of the basic mathematical approaches to study the properties of sets and spaces. The concept of fuzzy topological spaces has been introduced by Chang [10], laying the basis for the generalization of topological approaches into the context of fuzzy mathematics. Intuitionistic fuzzy topological spaces have been introduced by Coker [11], representing an extension of topological notions to intuitionistic fuzzy contexts. Olgun, Ünver, and Yardımcı [13] have proposed Pythagorean fuzzy topological spaces. Riaz et al. [14] have developed cubic m-polar fuzzy topology and explored its application in multi-criteria group decision-making problems.

Cubic Bipolar Pythagorean Fuzzy Set (CBPFS) is a set which incorporates the features of cubic, bipolar, and Pythagorean fuzzy sets in a single form. The set models uncertain data with the help of interval and exact memberships while taking positive and negative assessments at the same time. Nagalakshmi and Maragathavalli [15] have proposed the CBPFS and studied the application of the set in multimedia decision-making problems. On the other hand, the formulation and application of Cubic Bipolar Pythagorean Fuzzy Topology (CBPF Topology) still need to be explored.

Inspired by these advancements, in this study, CBPF Topology is developed in relation to the P-order. Some basic topological definitions like open set, closed set, interior, exterior, frontier, neighborhood, basis and subspace are formulated and examined in the context of this new proposed structure. The developed CBPF topology serves as a mathematical framework for modelling uncertain data in CBPFSs.

Moreover, in order to show the practicability of the proposed framework, a CRITIC-MAIRCA-based MCGDM method is developed in the context of CBPF set theory. This method is used in solving a clinical decision-making problem related to chronic cancer diagnosis by evaluating different treatment options against some uncertain clinical criteria. In addition, a comparative study is provided in order to reveal the validity and efficiency of the proposed method in comparison with the existing approaches.

II. PRELIMINARIES

Definition 2.1 [15]

Let $\mathscr{M}$ be a universe of discourse. A **cubic bipolar Pythagorean fuzzy set (CBPFS)** C_{BP} on universe set $\mathscr{M}$ is defined as follows

$$C_{BP} = \{(x, C(x), \sigma(x)) \mid x \in \mathcal{M}\}$$

where $C(x) = \{(x, [(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}), (\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}})], [(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}), (\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}})]) \mid x \in \mathcal{M}\}$ represents the IVBPFS defined on $\mathscr{M}$ while $\sigma(x) = \{(x, [\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}], [\mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}]) \mid x \in \mathcal{M}\}$ represents the BPFS defined on $\mathscr{M}$ in which the mappings $\mu^{+}_{C_{BP}} : \mathcal{M} \to [0, 1]$ denote the degree of positive membership, $\vartheta^{+}_{C_{BP}} : \mathcal{M} \to [0, 1]$ denote the degree of positive non-membership, $\mu^{-}_{C_{BP}} : \mathcal{M} \to [-1, 0]$ denote the degree of negative membership and $\vartheta^{-}_{C_{BP}} : \mathcal{M} \to [-1, 0]$ denote the degree of negative non-membership of each element $x \in \mathscr{M}$ to the set P_B such that $(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}) \subset [0,1]$, $(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}) \subset [0,1]$, $(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}) \subset [-1, 0]$, $(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}) \subset [-1, 0]$ and $\mu^{+l}_{C_{BP}} \leq \mu^{+u}_{C_{BP}}$, $\mu^{-l}_{C_{BP}} \leq \mu^{-u}_{C_{BP}}$, $\vartheta^{+l}_{C_{BP}} \leq \vartheta^{+u}_{C_{BP}}$ $\vartheta^{-l}_{C_{BP}} \leq \vartheta^{-u}_{C_{BP}}$.

Also, $0 \leq (\mu^{+u}_{C_{BP}})^2 + (\vartheta^{+u}_{C_{BP}})^2 \leq 1$, $0 \leq (\mu^{-u}_{C_{BP}})^2 + (\vartheta^{-u}_{C_{BP}})^2 \leq 1$.

$0 \leq (\mu^{+l}_{C_{BP}})^2 + (\vartheta^{+l}_{C_{BP}})^2 \leq 1$, $0 \leq (\mu^{-l}_{C_{BP}})^2 + (\vartheta^{-l}_{C_{BP}})^2 \leq 1$

and $0 \leq (\mu^{+}_{C_{BP}})^2 + (\vartheta^{+}_{C_{BP}})^2 \leq 1$, $0 \leq (\mu^{-}_{C_{BP}})^2 + (\vartheta^{-}_{C_{BP}})^2 \leq 1$.

We denote this pair $C_{BP} = (C, \sigma)$ where

$C = \left(\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right], \left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right]\right)$ and
$\sigma = \left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right]$ is known as cubic bipolar Pythagorean fuzzy number (CBPFN) .
Let $\pi_{C_{BP}}(x) = \left(\left[\pi^{+l}_{C_{BP}}(x), \pi^{+u}_{C_{BP}}(x)\right], \left[\pi^{-l}_{C_{BP}}(x), \pi^{-u}_{C_{BP}}(x)\right]; \left[\pi^{l}_{C_{BP}}(x), \pi^{u}_{C_{BP}}(x)\right]\right)$ for all x in $\mathscr{M}$, then π_{P_C} is called the CBPF index of x to $\mathscr{M}$, where

$$\pi^{+l}_{C_{BP}}(x) = \sqrt{1 - \left(\mu^{+l}_{C_{BP}}(x)\right)^2 - \left(\vartheta^{+l}_{C_{BP}}(x)\right)^2},\ \pi^{+u}_{C_{BP}}(x) = \sqrt{1 - \left(\mu^{+u}_{C_{BP}}(x)\right)^2 - \left(\vartheta^{+u}_{C_{BP}}(x)\right)^2},$$

$$\pi^{-l}_{C_{BP}}(x) = \sqrt{1 - \left(\mu^{-l}_{C_{BP}}(x)\right)^2 - \left(\vartheta^{-l}_{C_{BP}}(x)\right)^2},\ \pi^{+u}_{C_{BP}}(x) = \sqrt{1 - \left(\mu^{-u}_{C_{BP}}(x)\right)^2 - \left(\vartheta^{-u}_{C_{BP}}(x)\right)^2} \text{ and}$$

$$\pi^{l}_{C_{BP}}(x) = \sqrt{1 - \left(\mu^{l}_{C_{BP}}(x)\right)^2 - \left(\vartheta^{l}_{C_{BP}}(x)\right)^2},\ \pi^{u}_{C_{BP}}(x) = \sqrt{1 - \left(\mu^{u}_{C_{BP}}(x)\right)^2 - \left(\vartheta^{u}_{C_{BP}}(x)\right)^2}.$$

Definition 2.2 [15]

Let $C_{BP_1} = (C_1, \sigma_1)$ and $C_{BP_2} = (C_2, \sigma_2)$ be two CBPFSs defined over the universal set $\mathscr{M}$ in which

$C_1 = \left(\left[\left(\mu^{+l}_{C_{BP_1}}, \mu^{+u}_{C_{BP_1}}\right), \left(\vartheta^{+l}_{C_{BP_1}}, \vartheta^{+u}_{C_{BP_1}}\right)\right], \left[\left(\mu^{-l}_{C_{BP_1}}, \mu^{-u}_{C_{BP_1}}\right), \left(\vartheta^{-l}_{C_{BP_1}}, \vartheta^{-u}_{C_{BP_1}}\right)\right]\right)$,
$\sigma_1 = \left[\mu^{+}_{C_{BP_1}}, \vartheta^{+}_{C_{BP_1}}, \mu^{-}_{C_{BP_1}}, \vartheta^{-}_{C_{BP_1}}\right]$
$C_2 = \left(\left[\left(\mu^{+l}_{C_{BP_2}}, \mu^{+u}_{C_{BP_2}}\right), \left(\vartheta^{+l}_{C_{BP_2}}, \vartheta^{+u}_{C_{BP_2}}\right)\right], \left[\left(\mu^{-l}_{C_{BP_2}}, \mu^{-u}_{C_{BP_2}}\right), \left(\vartheta^{-l}_{C_{BP_2}}, \vartheta^{-u}_{C_{BP_2}}\right)\right]\right)$ and
$\sigma_2 = \left[\mu^{+}_{C_{BP_2}}, \vartheta^{+}_{C_{BP_2}}, \mu^{-}_{C_{BP_2}}, \vartheta^{-}_{C_{BP_2}}\right]$.

Then we define:

Equality $C_{BP_1} = C_{BP_2}$ if $\left(\mu^{+l}_{C_{BP_1}}, \mu^{+u}_{C_{BP_1}}\right) = \left(\mu^{+l}_{C_{BP_2}}, \mu^{+u}_{C_{BP_2}}\right)$, $\left(\vartheta^{+l}_{C_{BP_1}}, \vartheta^{+u}_{C_{BP_1}}\right) = \left(\vartheta^{+l}_{C_{BP_2}}, \vartheta^{+u}_{C_{BP_2}}\right)$, $\left(\mu^{-l}_{C_{BP_1}}, \mu^{-u}_{C_{BP_1}}\right) = \left(\mu^{-l}_{C_{BP_2}}, \mu^{-u}_{C_{BP_2}}\right)$, $\left(\vartheta^{-l}_{C_{BP_1}}, \vartheta^{-u}_{C_{BP_1}}\right) = \left(\vartheta^{-l}_{C_{BP_2}}, \vartheta^{-u}_{C_{BP_2}}\right)$, $\mu^{+}_{C_{BP_1}} = \mu^{+}_{C_{BP_2}}$, $\vartheta^{+}_{C_{BP_1}} = \vartheta^{+}_{C_{BP_2}}$, $\mu^{-}_{C_{BP_1}} = \mu^{-}_{C_{BP_2}}$ and $\vartheta^{-}_{C_{BP_1}} = \vartheta^{-}_{C_{BP_2}}$.

$\mathbb{P}$-order $C_{BP_1} \subseteq_{\mathbb{P}} C_{BP_2}$ if $\left(\mu^{+l}_{C_{BP_1}}, \mu^{+u}_{C_{BP_1}}\right) \subseteq \left(\mu^{+l}_{C_{BP_2}}, \mu^{+u}_{C_{BP_2}}\right)$, $\left(\vartheta^{+l}_{C_{BP_1}}, \vartheta^{+u}_{C_{BP_1}}\right) \supseteq \left(\vartheta^{+l}_{C_{BP_2}}, \vartheta^{+u}_{C_{BP_2}}\right)$, $\left(\mu^{-l}_{C_{BP_1}}, \mu^{-u}_{C_{BP_1}}\right) \supseteq \left(\mu^{-l}_{C_{BP_2}}, \mu^{-u}_{C_{BP_2}}\right)$, $\left(\vartheta^{-l}_{C_{BP_1}}, \vartheta^{-u}_{C_{BP_1}}\right) \subseteq \left(\vartheta^{-l}_{C_{BP_2}}, \vartheta^{-u}_{C_{BP_2}}\right)$, $\mu^{+}_{C_{BP_1}} \leq \mu^{+}_{C_{BP_2}}$, $\vartheta^{+}_{C_{BP_1}} \geq \vartheta^{+}_{C_{BP_2}}$, $\mu^{-}_{C_{BP_1}} \geq \mu^{-}_{C_{BP_2}}$ and $\vartheta^{-}_{C_{BP_1}} \leq \vartheta^{-}_{C_{BP_2}}$

Definition 2.3

Let $C_{BP_i} = (C_i, \sigma_i)$ where $i \in \Lambda$, be a family of CBPFSs defined over the set $\mathscr{M}$. Then we define:

(i) **$\mathbb{P}$-union:**

$$\bigcup_{\mathbb{P}} C_{BP_i}$$
$$= \left\{\left[\left(\sup_{i \in \Lambda} \mu^{+l}_{C_{BP_i}}, \sup_{i \in \Lambda} \mu^{+u}_{C_{BP_i}}\right), \left(\inf_{i \in \Lambda} \vartheta^{+l}_{C_{BP_i}}, \inf_{i \in \Lambda} \vartheta^{+u}_{C_{BP_i}}\right), \left(\inf_{i \in \Lambda} \mu^{-l}_{C_{BP_i}}, \inf_{i \in \Lambda} \mu^{-u}_{C_{BP_i}}\right), \left(\sup_{i \in \Lambda} \vartheta^{-l}_{C_{BP_i}}, \sup_{i \in \Lambda} \vartheta^{-u}_{C_{BP_i}}\right)\right],\right.$$
$$\left.\left[\sup_{i \in \Lambda} \mu^{+}_{C_{BP_i}}, \inf_{i \in \Lambda} \vartheta^{+}_{C_{BP_i}}, \inf_{i \in \Lambda} \mu^{-}_{C_{BP_i}}, \sup_{i \in \Lambda} \vartheta^{-}_{C_{BP_i}}\right]\right\}$$

(ii) **$\mathbb{P}$- intersection:**

$$\bigcap_{\mathbb{P}} C_{BP_i}$$
$$= \left\{\left[\left(\inf_{i\in\Lambda}\mu^{+l}_{C_{BP_i}}, \inf_{i\in\Lambda}\mu^{+u}_{C_{BP_i}}\right), \left(\sup_{i\in\Lambda}\vartheta^{+l}_{C_{BP_i}}, \sup_{i\in\Lambda}\vartheta^{+u}_{C_{BP_i}}\right), \left(\sup_{i\in\Lambda}\mu^{-l}_{C_{BP_i}}, \sup_{i\in\Lambda}\mu^{-u}_{C_{BP_i}}\right), \left(\inf_{i\in\Lambda}\vartheta^{-l}_{C_{BP_i}}, \inf_{i\in\Lambda}\vartheta^{-u}_{C_{BP_i}}\right)\right], \left[\inf_{i\in\Lambda}\mu^{+}_{C_{BP_i}}, \sup_{i\in\Lambda}\vartheta^{+}_{C_{BP_i}}, \sup_{i\in\Lambda}\mu^{-}_{C_{BP_i}}, \inf_{i\in\Lambda}\vartheta^{-}_{C_{BP_i}}\right]\right\}$$

Definition 2.4 [15]

Let $C_{BP} = (C, \sigma)$ in which $C = \left(\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right], \left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right]\right)$ and $\sigma = \left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right]$ be a CBPFS defined over the universe $\mathscr{M}$. Then, the **complement** of C_{BP} can be defined as $(C_{BP})^C = (C^C, \sigma^C)$ where $C^C = \left(\left[\left(1-\mu^{+u}_{C_{BP}}, 1-\mu^{+l}_{C_{BP}}\right), \left(1-\vartheta^{+u}_{C_{BP}}, 1-\vartheta^{+l}_{C_{BP}}\right)\right], \left[\left(-1-\mu^{-u}_{C_{BP}}, -1-\mu^{-l}_{C_{BP}}\right), \left(-1-\vartheta^{-u}_{C_{BP}}, -1-\vartheta^{-l}_{C_{BP}}\right)\right]\right)$ and $\sigma^C = \left[1-\mu^{+}_{C_{BP}}, 1-\vartheta^{+}_{C_{BP}}, -1-\mu^{-}_{C_{BP}}, -1-\vartheta^{-}_{C_{BP}}\right]$.

Theorem 2.5 [15]

Let $C_{BP_1} = (C_1, \sigma_1)$ and $C_{BP_2} = (C_2, \sigma_2)$ be two CBPFSs defined over the universal set $\mathscr{M}$. Then the following properties are hold under $\mathbb{P}$-order:

(i) $C_{BP_1} \cup_{\mathbb{P}} C_{BP_2} = C_{BP_2} \cup_{\mathbb{P}} C_{BP_1}$

(ii) $C_{BP_1} \cap_{\mathbb{P}} C_{BP_2} = C_{BP_2} \cap_{\mathbb{P}} C_{BP_1}$

(iii) $\left(C_{BP_1} \cup_{\mathbb{P}} C_{BP_2}\right)^C = \left(C_{BP_1}\right)^C \cap_{\mathbb{P}} \left(C_{BP_2}\right)^C$

(iv) $\left(C_{BP_1} \cap_{\mathbb{P}} C_{BP_2}\right)^C = \left(C_{BP_1}\right)^C \cup_{\mathbb{P}} \left(C_{BP_2}\right)^C$.

Proof: Straightforward.

Definition 2.6 [15]

The score function of a CBPFN $C_{BP} = (C, \sigma)$, in which $C = \left(\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right], \left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right]\right)$ and $\sigma = \left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right]$ is defined as follows:

$$S_C(C_{BP}) = \left(\left(\frac{\mu^{+l}_{C_{BP}}+\mu^{+u}_{C_{BP}}-\mu^{+}_{C_{BP}}}{3}\right)^2 - \left(\frac{\vartheta^{+l}_{C_{BP}}+\vartheta^{+u}_{C_{BP}}-\vartheta^{+}_{C_{BP}}}{3}\right)^2 + \left(\frac{\mu^{-l}_{C_{BP}}+\mu^{-u}_{C_{BP}}-\mu^{-}_{C_{BP}}}{3}\right)^2 - \left(\frac{\vartheta^{-l}_{C_{BP}}+\vartheta^{-u}_{C_{BP}}-\vartheta^{-}_{C_{BP}}}{3}\right)^2\right),$$

where $0 \leq S_C(C_{BP}) \leq 1$

Definition 2.7 [15]

Let $C_{BP} = (C, \sigma)$ be a CBPFN, where $C = \left(\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right], \left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right]\right)$ and $\sigma = \left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right]$.

Then the accuracy function of C_{BP} is defined as follows:

$$A_C(C_{BP}) = \left(\left(\frac{\mu^{+l}_{C_{BP}}+\mu^{+u}_{C_{BP}}-\mu^{+}_{C_{BP}}}{3}\right)^2 + \left(\frac{\vartheta^{+l}_{C_{BP}}+\vartheta^{+u}_{C_{BP}}-\vartheta^{+}_{C_{BP}}}{3}\right)^2 + \left(\frac{\mu^{-l}_{C_{BP}}+\mu^{-u}_{C_{BP}}-\mu^{-}_{C_{BP}}}{3}\right)^2 + \left(\frac{\vartheta^{-l}_{C_{BP}}+\vartheta^{-u}_{C_{BP}}-\vartheta^{-}_{C_{BP}}}{3}\right)^2\right),$$

where $-1 \leq A_C(C_{BP}) \leq 1$.

III. Cubic Bipolar Pythagorean Fuzzy Topology under ℙ-order

In this section, we introduce the concept of a ℙ-cubic bipolar Pythagorean fuzzy topology (ℙ-CBPFT) or a cubic bipolar Pythagorean fuzzy topology with ℙ-order.

Definition 3.1

A CBPFS $C_{BP} = (C, \sigma)$ where $C = \left(\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right], \left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right]\right)$ and $\sigma = \left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right]$ for which $\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right]$=[(0,0),(1,1)] , $\left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right] = [(-1,-1),(0,0)]$ and $\left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right] = [1,0,-1,0]$ is denoted by ${}^{ll}C_{BP}$

Definition 3.2

A CBPFS $C_{BP} = (C, \sigma)$ where $C = \left(\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right], \left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right]\right)$ and $\sigma = \left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right]$ for which $\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right]$=[(0,0),(1,1)] , $\left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right] = [(0,0),(-1,-1)]$ and $\left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right] = [1,0,-1,0]$ is denoted by ${}^{lu}C_{BP}$

Definition 3.3

A CBPFS $C_{BP} = (C, \sigma)$ where $C = \left(\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right], \left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right]\right)$ and $\sigma = \left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right]$ for which $\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right]$=[(1,1),(0,0)] , $\left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right] = [(-1,-1),(0,0)]$and $\left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right] = [1,0,-1,0]$ is denoted by ${}^{ul}C_{BP}$.

Definition 3.4

A CBPFS $C_{BP} = (C, \sigma)$ where $C = \left(\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right], \left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right]\right)$ and $\sigma = \left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right]$ for which $\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right]$=[(1,1),(0,0)] , $\left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right] = [(0,0),(-1,-1)]$ and $\left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right] = [1,0,-1,0]$ is denoted by ${}^{uu}C_{BP}$.

Definition 3.5

A CBPFS $C_{BP} = (C, \sigma)$ where $C = \left(\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right], \left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right]\right)$ and $\sigma = \left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right]$ for which $\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right]$=[(1,1),(0,0)] , $\left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right] = [(0,0),(-1,-1)]$ and $\left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right] = [0,1,0,-1]$ is denoted by ${}^{\overline{ll}}C_{BP}$

Definition 3.6

A CBPFS $C_{BP} = (C, \sigma)$ where $C = \left(\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right], \left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right]\right)$ and $\sigma = \left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right]$ for which $\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right]$=[(1,1),(0,0)] , $\left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right] = [(-1,-1),(0,0)]$and $\left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right] = [0,1,0,-1]$ is denoted by ${}^{\overline{lu}}C_{BP}$

Definition 3.7

A CBPFS $C_{BP} = (C, \sigma)$ where $C = \left(\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right], \left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right]\right)$ and $\sigma = \left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right]$ for which $\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right]$=[(0,0),(1,1)] , $\left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right] = [(0,0), (-1,-1)]$ and $\left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right] = [0,1,0,-1]$ is denoted by ${}^{\overline{ul}}C_{BP}$.

Definition 3.8

A CBPFS $C_{BP} = (C, \sigma)$ where $C = \left(\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right], \left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right]\right)$ and $\sigma = \left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right]$ for which $\left[\left(\mu^{+l}_{C_{BP}}, \mu^{+u}_{C_{BP}}\right), \left(\vartheta^{+l}_{C_{BP}}, \vartheta^{+u}_{C_{BP}}\right)\right]$=[(0,0),(1,1)] , $\left[\left(\mu^{-l}_{C_{BP}}, \mu^{-u}_{C_{BP}}\right), \left(\vartheta^{-l}_{C_{BP}}, \vartheta^{-u}_{C_{BP}}\right)\right] = [(-1,-1), (0,0)]$ and $\left[\mu^{+}_{C_{BP}}, \vartheta^{+}_{C_{BP}}, \mu^{-}_{C_{BP}}, \vartheta^{-}_{C_{BP}}\right] = [0,1,0,-1]$ is denoted by ${}^{\overline{uu}}C_{BP}$.

Definition 3.9

Let $\mathscr{M}$ be a non-empty universal set and $\mathbb{C}_{BP}(\mathcal{M})$ to be the accumulation of all CBPFSs in $\mathscr{M}$. If the collection $\tau_{\mathbb{P}C_{BP}}$ containing CBPFSs satisfies the following conditions, it is termed as a **$\mathbb{P}$-cubic bipolar Pythagorean fuzzy topology ($\mathbb{P}$-CBPFT),**

(i) ${}^{ll}C_{BP}$, ${}^{lu}C_{BP}$, ${}^{ul}C_{BP}$, ${}^{uu}C_{BP}$, ${}^{\overline{ll}}C_{BP}$, ${}^{\overline{lu}}C_{BP}$, ${}^{\overline{ul}}C_{BP}$ and ${}^{\overline{uu}}C_{BP}$ $\in \tau_{\mathbb{P}C_{BP}}$.

(ii) If $(\mathbb{P}C_{BP})_i \in \tau_{\mathbb{P}C_{BP}} \forall i$ then $\cup_{\mathbb{P}}(\mathbb{P}C_{BP})_i \in \tau_{\mathbb{P}C_{BP}}$

(iii) If $\mathbb{P}C_{BP_1}, \mathbb{P}C_{BP_2} \in \tau_{\mathbb{P}C_{BP}}$ then $\mathbb{P}C_{BP_1} \cap_{\mathbb{P}} \mathbb{P}C_{BP_2} \in \tau_{\mathbb{P}C_{BP}}$.

Then, the pair ($\mathscr{M}$, $\tau_{\mathbb{P}C_{BP}}$) is called $\mathbb{P}$-cubic bipolar Pythagorean fuzzy topological space.

Example 3.10

Let $\mathscr{M}$ be a universal set and $\mathbb{P}\mathbb{C}_{BP}(\mathcal{M})$ be the assemblage of all $\mathbb{P}$-cubic bipolar Pythagorean fuzzy sets ($\mathbb{P}$-CBPFS) in $\mathscr{M}$. Consider $\mathbb{P}$-CBPF subsets of $\mathbb{P}\mathbb{C}_{BP}(\mathcal{M})$ given as

$\mathbb{P}C_{BP_1}$ = {[(0, 0), (1, 1)], [(0, 0), (-1, -1)], [0.21, 0.32, -0.31, -0.45]}

$\mathbb{P}C_{BP_2}$ = {[(1, 1), (0, 0)], [(0, 0), (-1, -1)], [0.21, 0.32, -0.31, -0.45]}

$\mathbb{P}C_{BP_3}$ = {[(0, 0), (1, 1)], [(-1, -1), (0, 0)], [0.21, 0.32, -0.31, -0.45]}

$\mathbb{P}C_{BP_4}$ = {[(1, 1), (0, 0)], [(-1, -1), (0, 0)], [0.21, 0.32, -0.31, -0.45]}

The union and intersection with a $\mathbb{P}$-order for the above $\mathbb{P}$-CBPFS are given in tables 1 and 2 respectively.

Table 1 $\mathbb{P}$-Union of $\mathbb{P}$-CBPFS

$\cup_{\mathbb{P}}$	${}^{ll}C_{BP}$	${}^{lu}C_{BP}$	${}^{ul}C_{BP}$	${}^{uu}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{lu}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_4}$
${}^{ll}C_{BP}$	${}^{ll}C_{BP}$	${}^{ll}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ll}C_{BP}$	${}^{ll}C_{BP}$	${}^{ll}C_{BP}$	${}^{ul}C_{BP}$	${}^{ll}C_{BP}$	${}^{ul}C_{BP}$
${}^{lu}C_{BP}$	${}^{ll}C_{BP}$	${}^{lu}C_{BP}$	${}^{ul}C_{BP}$	${}^{uu}C_{BP}$	${}^{uu}C_{BP}$	${}^{ul}C_{BP}$	${}^{lu}C_{BP}$	${}^{ll}C_{BP}$	${}^{lu}C_{BP}$	${}^{uu}C_{BP}$	${}^{ll}C_{BP}$	${}^{ul}C_{BP}$
${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$
${}^{uu}C_{BP}$	${}^{ul}C_{BP}$	${}^{uu}C_{BP}$	${}^{ul}C_{BP}$	${}^{uu}C_{BP}$	${}^{uu}C_{BP}$	${}^{ul}C_{BP}$	${}^{uu}C_{BP}$	${}^{ul}C_{BP}$	${}^{uu}C_{BP}$	${}^{uu}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$
${}^{\overline{ll}}C_{BP}$	${}^{ul}C_{BP}$	${}^{uu}C_{BP}$	${}^{ul}C_{BP}$	${}^{uu}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{lu}}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{lu}}C_{BP}$	$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$
${}^{\overline{lu}}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{\overline{lu}}C_{BP}$	${}^{\overline{lu}}C_{BP}$	${}^{\overline{lu}}C_{BP}$	${}^{\overline{lu}}C_{BP}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$

${}^{\overline{ul}}C_{BP}$	${}^{ll}C_{BP}$	${}^{lu}C_{BP}$	${}^{ul}C_{BP}$	${}^{uu}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{lu}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_4}$
${}^{\overline{uu}}C_{BP}$	${}^{ll}C_{BP}$	${}^{ll}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{\overline{lu}}C_{BP}$	${}^{\overline{lu}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_4}$
$\mathbb{P}C_{BP_1}$	${}^{ll}C_{BP}$	${}^{lu}C_{BP}$	${}^{ul}C_{BP}$	${}^{uu}C_{BP}$	$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_4}$
$\mathbb{P}C_{BP_2}$	${}^{ul}C_{BP}$	${}^{uu}C_{BP}$	${}^{ul}C_{BP}$	${}^{uu}C_{BP}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$
$\mathbb{P}C_{BP_3}$	${}^{ll}C_{BP}$	${}^{ll}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_4}$
$\mathbb{P}C_{BP_4}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	${}^{ul}C_{BP}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_4}$

Table 2 $\mathbb{P}$-Intersection of $\mathbb{P}$-CBPFS

$\cap_{\mathbb{P}}$	${}^{ll}C_{BP}$	${}^{lu}C_{BP}$	${}^{ul}C_{BP}$	${}^{uu}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{lu}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_4}$
${}^{ll}C_{BP}$	${}^{ll}C_{BP}$	${}^{lu}C_{BP}$	${}^{ll}C_{BP}$	${}^{lu}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_3}$
${}^{lu}C_{BP}$	${}^{lu}C_{BP}$	${}^{lu}C_{BP}$	${}^{lu}C_{BP}$	${}^{lu}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_1}$
${}^{ul}C_{BP}$	${}^{ll}C_{BP}$	${}^{lu}C_{BP}$	${}^{ul}C_{BP}$	${}^{uu}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{lu}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_4}$
${}^{uu}C_{BP}$	${}^{lu}C_{BP}$	${}^{lu}C_{BP}$	${}^{uu}C_{BP}$	${}^{uu}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_2}$
${}^{\overline{ll}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ll}}C_{BP}$
${}^{\overline{lu}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{lu}}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{lu}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	${}^{\overline{lu}}C_{BP}$
${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$
${}^{\overline{uu}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	${}^{\overline{uu}}C_{BP}$
$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_1}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_1}$
$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_2}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_2}$
$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_1}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_3}$
$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_4}$	$\mathbb{P}C_{BP_2}$	${}^{\overline{ll}}C_{BP}$	${}^{\overline{lu}}C_{BP}$	${}^{\overline{ul}}C_{BP}$	${}^{\overline{uu}}C_{BP}$	$\mathbb{P}C_{BP_1}$	$\mathbb{P}C_{BP_2}$	$\mathbb{P}C_{BP_3}$	$\mathbb{P}C_{BP_4}$

Clearly, $\tau^{1}_{\mathbb{P}C_{BP}} = \left\{ {}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, {}^{\overline{ll}}C_{BP}, {}^{\overline{lu}}C_{BP}, {}^{\overline{ul}}C_{BP}, {}^{\overline{uu}}C_{BP} \right\}$ and $\tau^{2}_{\mathbb{P}C_{BP}} = \left\{ {}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, {}^{\overline{ll}}C_{BP}, {}^{\overline{lu}}C_{BP}, {}^{\overline{ul}}C_{BP}, {}^{\overline{uu}}C_{BP}, \mathbb{P}C_{BP_1}, \mathbb{P}C_{BP_2}, \mathbb{P}C_{BP_3}, \mathbb{P}C_{BP_4} \right\}$ are cubic bipolar Pythagorean fuzzy topology with $\mathbb{P}$-order.

Definition 3.11

Let $\mathscr{M}$ be a non-empty universal set and $\tau_{\mathbb{P}C_{BP}} = \{(\mathbb{P}C_{BP})_i / i \in \Lambda\}$ where $(\mathbb{P}C_{BP})_i$ represent the $\mathbb{P}$-cubic bipolar Pythagorean fuzzy subsets of universal set $\mathscr{M}$.

Then, $\tau_{\mathbb{P}C_{BP}}$ is termed as a $\mathbb{P}$-cubic bipolar Pythagorean fuzzy topology on $\mathscr{M}$ and it is the largest $\mathbb{P}$-cubic bipolar Pythagorean fuzzy topology on $\mathscr{M}$ and is called as $\mathbb{P}$-discrete cubic bipolar Pythagorean fuzzy topology.

Definition 3.12

Let $\mathscr{M}$ be a non-empty universal set and $\tau_{\mathbb{P}C_{BP}} = \{ {}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, {}^{\overline{ll}}C_{BP}, {}^{\overline{lu}}C_{BP}, {}^{\overline{ul}}C_{BP}, {}^{\overline{uu}}C_{BP} \}$ be the collection of $\mathbb{P}$-cubic bipolar Pythagorean fuzzy subsets.

Then, $\tau_{\mathbb{P}C_{BP}}$ is termed as a $\mathbb{P}$-cubic bipolar Pythagorean fuzzy topology on $\mathscr{M}$ and it is the smallest $\mathbb{P}$-cubic bipolar Pythagorean fuzzy topology on $\mathscr{M}$ and is called as $\mathbb{P}$-indiscrete cubic bipolar Pythagorean fuzzy topology.

Definition 3.13

The elements of $\mathbb{P}$-cubic bipolar Pythagorean fuzzy topology $\tau_{\mathbb{P}C_{BP}}$ is termed as $\mathbb{P}$ **-cubic bipolar Pythagorean fuzzy open sets** $\mathbb{P}$ **-CBPFOSs** in $(\mathscr{M}, \tau_{\mathbb{P}C_{BP}})$.

Theorem 3.14

If $(\mathscr{M}, \tau_{\mathbb{P}C_{BP}})$ is any $\mathbb{P}$-cubic bipolar Pythagorean fuzzy topological space. Then,

(i) ${}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, {}^{\overline{ll}}C_{BP}, {}^{\overline{lu}}C_{BP}, {}^{\overline{ul}}C_{BP}$ and ${}^{\overline{uu}}C_{BP}$ are $\mathbb{P}$-CBPFOSs

(ii) The $\mathbb{P}$-union of any number of $\mathbb{P}$-CBPFOSs is $\mathbb{P}$-CBPFOS.

(iii) The $\mathbb{P}$-intersection of finite number of $\mathbb{P}$-CBPFOSs is $\mathbb{P}$-CBPFOS.

Proof:

(i) From the definition 3.9 of $\mathbb{P}$-CBPFT, ${}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, {}^{\overline{ll}}C_{BP}, {}^{\overline{lu}}C_{BP}, {}^{\overline{ul}}C_{BP}$ and ${}^{\overline{uu}}C_{BP} \in \tau_{\mathbb{P}C_{BP}}$. Hence, ${}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, {}^{\overline{ll}}C_{BP}, {}^{\overline{lu}}C_{BP}, {}^{\overline{ul}}C_{BP}$ and ${}^{\overline{uu}}C_{BP}$ are $\mathbb{P}$-CBPFOSs.

(ii) Let $\{(\mathbb{P}C_{BP})_i / i \in \Lambda\}$ be $\mathbb{P}$-CBPFOSs. Then, from the definition of $\mathbb{P}$-CBPFT,

$$\bigcup_{\mathbb{P}} (\mathbb{P}C_{BP})_i \in \tau_{\mathbb{P}C_{BP}} .$$

$$\text{Hence, } \bigcup_{\mathbb{P}} (\mathbb{P}C_{BP})_i \text{ is } \mathbb{P} - \text{CBPFOS.}$$

(iii) Let $\mathbb{P}C_{BP_1}, \mathbb{P}C_{BP_2}, \ldots, \mathbb{P}C_{BP_n}$ be $\mathbb{P}$-CBPFOSs. Then, from the definition of $\mathbb{P}$-CBPFT,

$$\bigcap_{\substack{i=1 \\ \mathbb{P}}}^{n} \mathbb{P}C_{BP_i} \in \tau_{\mathbb{P}C_{BP}}.$$

$$\text{Hence, } \bigcap_{\substack{i=1 \\ \mathbb{P}}}^{n} \mathbb{P}C_{BP_i} \text{ is } \mathbb{P} - \text{CBPFOS.}$$

Definition 3.15

The complement of elements of $\mathbb{P}$ -cubic bipolar Pythagorean fuzzy open sets is termed as $\mathbb{P}$ **-cubic bipolar Pythagorean fuzzy closed sets** $\mathbb{P}$ **-CBPFCSs** in $(\mathscr{M}, \tau_{\mathbb{P}C_{BP}})$.

Theorem 3.16

If $(\mathscr{M}, \tau_{\mathbb{P}C_{BP}})$ is any $\mathbb{P}$-cubic bipolar Pythagorean fuzzy topological space. Then,

(i) ${}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, \overline{{}^{ll}}C_{BP}, \overline{{}^{lu}}C_{BP}, \overline{{}^{ul}}C_{BP}$ and $\overline{{}^{uu}}C_{BP}$ are ℙ-CBPFCSs

(ii) The ℙ-intersection of any number of ℙ-CBPFCSs is ℙ-CBPFCS.

(iii) The ℙ-union of finite number of ℙ-CBPFCSs is ℙ-CBPFCS.

Proof:

(i) ${}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, \overline{{}^{ll}}C_{BP}, \overline{{}^{lu}}C_{BP}, \overline{{}^{ul}}C_{BP}$ and $\overline{{}^{uu}}C_{BP}$ are ℙ-CBPFOSs . Since the complement of ${}^{ll}C_{BP} = \overline{{}^{ll}}C_{BP}$, ${}^{lu}C_{BP} = \overline{{}^{lu}}C_{BP}$, ${}^{ul}C_{BP} = \overline{{}^{ul}}C_{BP}$ and ${}^{uu}C_{BP} = \overline{{}^{uu}}C_{BP}$. From the definition 3.9 of ℙ-CBPFT,

${}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, \overline{{}^{ll}}C_{BP}, \overline{{}^{lu}}C_{BP}, \overline{{}^{ul}}C_{BP}$ and $\overline{{}^{uu}}C_{BP} \in \tau_{\mathbb{P}C_{BP}}$.

So, ${}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, \overline{{}^{ll}}C_{BP}, \overline{{}^{lu}}C_{BP}, \overline{{}^{ul}}C_{BP}$ and $\overline{{}^{uu}}C_{BP}$ are ℙ-CBPFCSs.

(ii) Let $\{(\mathbb{P}C_{BP})_i / i \in \Lambda\}$ be ℙ-CBPFCSs. Then, $\overline{(\mathbb{P}C_{BP})_\imath} \in \tau_{\mathbb{P}C_{BP}}$.

From the definition of ℙ-CBPFT,

$$\bigcup_{\mathbb{P}} \overline{(\mathbb{P}C_{BP})_\imath} \in \tau_{\mathbb{P}C_{BP}} .$$

$$\text{Hence} \bigcup_{\mathbb{P}} \overline{(\mathbb{P}C_{BP})_\imath} \text{ is } \mathbb{P}-\text{CBPFOS}, \text{but} \bigcup_{\mathbb{P}} \overline{(\mathbb{P}C_{BP})_\imath} = \overline{\bigcap_{\mathbb{P}} (\mathbb{P}C_{BP})_\imath}.$$

$$\text{So,} \bigcap_{\mathbb{P}} (\mathbb{P}C_{BP})_i \text{ is } \mathbb{P}-\text{CBPFCS}$$

(iii) Let $\mathbb{P}C_{BP_1}, \mathbb{P}C_{BP_2}, \ldots, \mathbb{P}C_{BP_n}$ be ℙ-CBPFOSs. Then, $\overline{\mathbb{P}C_{BP_1}}, \overline{\mathbb{P}C_{BP_2}}, \ldots, \overline{\mathbb{P}C_{BP_n}}$ are ℙ-CBPFCSs. From the definition of ℙ-CBPFT,

$$\bigcap_{\substack{i=1\\ \mathbb{P}}}^{n} \overline{(\mathbb{P}C_{BP})_\imath} \in \tau_{\mathbb{P}C_{BP}}.$$

$$\text{Hence} \bigcap_{\substack{i=1\\ \mathbb{P}}}^{n} \overline{(\mathbb{P}C_{BP})_\imath} \text{ is } \mathbb{P}-\text{CBPFOS}, \text{but} \bigcap_{\substack{i=1\\ \mathbb{P}}}^{n} \overline{(\mathbb{P}C_{BP})_\imath} = \overline{\bigcup_{\substack{\imath=1\\ \mathbb{P}}}^{n} (\mathbb{P}C_{BP})_\imath}$$

$$\text{So,} \bigcup_{\substack{i=1\\ \mathbb{P}}}^{n} (\mathbb{P}C_{BP})_i \text{ is } \mathbb{P}-\text{CBPFCS}$$

Definition 3.17

The ℙ-cubic bipolar Pythagorean fuzzy sets which are ℙ-CBPFOSs and ℙ-CBPFCSs, are termed as ℙ-cubic bipolar Pythagorean fuzzy clopen sets in $(\mathscr{M}, \tau_{\mathbb{P}C_{BP}})$.

Proposition 3.18

(i) For every $\tau_{\mathbb{P}C_{BP}}$, ${}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, \overline{{}^{ll}}C_{BP}, \overline{{}^{lu}}C_{BP}, \overline{{}^{ul}}C_{BP}$ and $\overline{{}^{uu}}C_{BP}$ are ℙ- cubic bipolar Pythagorean fuzzy clopen sets.

(ii) For discrete ℙ- cubic bipolar Pythagorean fuzzy topology, all the ℙ- cubic bipolar Pythagorean fuzzy subsets of $\mathscr{M}$ are ℙ- cubic bipolar Pythagorean fuzzy clopen sets.

(iii) For indiscrete ℙ- cubic bipolar Pythagorean fuzzy topology, ${}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, \overline{{}^{ll}}C_{BP}, \overline{{}^{lu}}C_{BP}$,

$\overline{{}^{ul}}C_{BP}$ and $\overline{{}^{uu}}C_{BP}$ are only ℙ- cubic bipolar Pythagorean fuzzy clopen sets.

Definition 3.19

Let ($\mathscr{M}$, $\tau^1_{\mathbb{P}C_{BP}}$) and ($\mathscr{M}$, $\tau^2_{\mathbb{P}C_{BP}}$)be two $\mathbb{P}$-CBPFTs in $\mathscr{M}$. Two $\mathbb{P}$-CBPFTs are said to be **comparable** if $\tau^1_{\mathbb{P}C_{BP}} \subseteq_{\mathbb{P}} \tau^2_{\mathbb{P}C_{BP}}$ or $\tau^2_{\mathbb{P}C_{BP}} \subseteq_{\mathbb{P}} \tau^1_{\mathbb{P}C_{BP}}$.

If $\tau^1_{\mathbb{P}C_{BP}} \subseteq_{\mathbb{P}} \tau^2_{\mathbb{P}C_{BP}}$, then $\tau^1_{\mathbb{P}C_{BP}}$ is called $\mathbb{P}$- cubic bipolar Pythagorean fuzzy coarser than $\tau^2_{\mathbb{P}C_{BP}}$ and $\tau^2_{\mathbb{P}C_{BP}}$ is called $\mathbb{P}$- cubic bipolar Pythagorean fuzzy finer than $\tau^1_{\mathbb{P}C_{BP}}$.

Example 3.20

Let $\mathscr{M}$ be a universal set and from Example 3.10

$$\tau^1_{\mathbb{P}C_{BP}} = \left\{ {}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, {}^{\overline{ll}}C_{BP}, {}^{\overline{lu}}C_{BP}, {}^{\overline{ul}}C_{BP}, {}^{\overline{uu}}C_{BP} \right\} \text{ and}$$

$$\tau^2_{\mathbb{P}C_{BP}} = \left\{ {}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, {}^{\overline{ll}}C_{BP}, {}^{\overline{lu}}C_{BP}, {}^{\overline{ul}}C_{BP}, {}^{\overline{uu}}C_{BP}, \mathbb{P}C_{BP_1}, \mathbb{P}C_{BP_2}, \mathbb{P}C_{BP_3}, \mathbb{P}C_{BP_4} \right\}$$

are $\mathbb{P}$- cubic bipolar Pythagorean fuzzy topologies on $\mathscr{M}$. Then, $\tau^1_{\mathbb{P}C_{BP}} \subseteq_{\mathbb{P}} \tau^2_{\mathbb{P}C_{BP}}$. Hence, $\tau^1_{\mathbb{P}C_{BP}}$ is called a $\mathbb{P}$- cubic bipolar Pythagorean fuzzy coarser than $\tau^2_{\mathbb{P}C_{BP}}$.

3.1 Subspace of $\mathbb{P}$ - CBPFT

Definition 3.21

Let ($\mathscr{M}$, $\tau_{\mathbb{P}C_{BP}\mathcal{M}}$) be a $\mathbb{P}$-CBPFT. If $\mathbb{Y} \subseteq \mathscr{M}$, then the collection

$\tau_{\mathbb{P}C_{BP}\mathbb{Y}} = \{ \mathbb{P}C_{BP}(\mathcal{M}) \cap_{\mathbb{P}} \mathbb{Y} \,/\, \mathbb{P}C_{BP}(\mathcal{M}) \in \tau_{\mathbb{P}C_{BP}\mathcal{M}} \}$

is a topology on $\mathbb{Y}$ and is called as the **$\mathbb{P}$-cubic bipolar Pythagorean fuzzy subspace topology.** Then, ($\mathbb{Y}$, $\tau_{\mathbb{P}C_{BP}\mathbb{Y}}$) is called a **$\mathbb{P}$-cubic bipolar Pythagorean fuzzy subspace of** ($\mathscr{M}$, $\tau_{\mathbb{P}C_{BP}\mathcal{M}}$).

Example 3.22

Let $\mathscr{M}$ be a nonempty set. From Example 3.10,

$$\tau^2_{\mathbb{P}C_{BP}} = \left\{ {}^{ll}C_{BP}, {}^{lu}C_{BP}, {}^{ul}C_{BP}, {}^{uu}C_{BP}, {}^{\overline{ll}}C_{BP}, {}^{\overline{lu}}C_{BP}, {}^{\overline{ul}}C_{BP}, {}^{\overline{uu}}C_{BP}, \mathbb{P}C_{BP_1}, \mathbb{P}C_{BP_2}, \mathbb{P}C_{BP_3}, \mathbb{P}C_{BP_4} \right\}$$

is a $\mathbb{P}$- cubic bipolar Pythagorean fuzzy topology on $\mathscr{M}$.

Now, consider any $\mathbb{P}$- cubic bipolar Pythagorean fuzzy set on $\mathscr{M}$ Such that $\mathbb{Y} \subseteq \mathscr{M}$ is

$\mathbb{Y}$ = {[(0.21, 0.33), (0.47, 0.52)], [(-0.13, -0.27), (-0.62, -0.36)], [0.18, 0.29, -0.14, -0.34]}

Since,

$\mathbb{Y} \cap_{\mathbb{P}} {}^{ll}C_{BP}$ = {[(0, 0), (1, 1)], [(-0.13, -0.27), (-0.62, -0.36)], [0.18, 0.29, -0.14, -0.34]}
= $\mathbb{P}C^1_{BP\mathbb{Y}}$

$\mathbb{Y} \cap_{\mathbb{P}} {}^{lu}C_{BP}$ = {[(0, 0), (1, 1)], [(0, 0), (-1, -1)], [0.18, 0.29, -0.14, -0.34]}
= $\mathbb{P}C^2_{BP\mathbb{Y}}$

$\mathbb{Y} \cap_{\mathbb{P}} {}^{ul}C_{BP}$ = [(0.21, 0.33), (0.47, 0.52)], [(-0.13, -0.27), (-0.62, -0.36)], [0.18, 0.29, -0.14, -0.34]}
= $\mathbb{Y}$

$\mathbb{Y} \cap_{\mathbb{P}} {}^{uu}C_{BP}$ = {[(0.21, 0.33), (0.47, 0.52)], [(0, 0), (-1, -1)], [0.18, 0.29, -0.14, -0.34]}
= $\mathbb{P}C^3_{BP\mathbb{Y}}$

$\mathbb{Y} \cap_{\mathbb{P}} {}^{\overline{ll}}C_{BP}$ = {[(0.21, 0.33), (0.47, 0.52)], [(0, 0), (-1, -1)], [0, 1, 0, -1]}
= $\mathbb{P}C^4_{BP\mathbb{Y}}$

$\mathbb{Y} \cap_{\mathbb{P}} {}^{\overline{lu}}C_{BP}$ = {[(0.21, 0.33), (0.47, 0.52)], [(-0.13, -0.27), (-0.62, -0.36)], [0, 1, 0, -1]}
= $\mathbb{P}C^5_{BP\mathbb{Y}}$

$\mathbb{Y} \cap_{\mathbb{P}} {}^{\overline{ul}}C_{BP}$ = {[(0, 0), (1, 1)], [(0, 0), (-1, -1)], [0, 1, 0, -1]}

$= \overline{^{ul}}C_{BP}$

$\mathbb{Y} \cap_{\mathbb{P}} \overline{^{uu}}C_{BP}$ = {[(0, 0), (1, 1)], [(-0.13, -0.27), (-0.62, -0.36)], [0, 1, 0, -1]}

$= \mathbb{P}C^{6}_{BP\mathbb{Y}}$

$\mathbb{Y} \cap_{\mathbb{P}} \mathbb{P}C_{BP_1}$ = {[(0, 0), (1, 1)], [(0, 0), (-1, -1)], [0.18, 0.32, -0.14, -0.34]}

$= \mathbb{P}C^{7}_{BP\mathbb{Y}}$

$\mathbb{Y} \cap_{\mathbb{P}} \mathbb{P}C_{BP_2}$ = {[(0.21, 0.33), (0.47, 0.52)], [(0, 0), (-1, -1)], [0.18, 0.29, -0.14, -0.45]}

$= \mathbb{P}C^{8}_{BP\mathbb{Y}}$

$\mathbb{Y} \cap_{\mathbb{P}} \mathbb{P}C_{BP_3}$ = {[(0, 0), (1, 1)], [(-0.13, -0.27), (-0.62, -0.36)], [0.18, 0.29, -0.14, -0.45]}

$= \mathbb{P}C^{9}_{BP\mathbb{Y}}$

$\mathbb{Y} \cap_{\mathbb{P}} \mathbb{P}C_{BP_4}$ = {[(0.21, 0.33), (0.47, 0.52)], [(-0.13, -0.27), (-0.62, -0.36)], [0.18, 0.29, -0.14, -0.45]}

$= \mathbb{P}C^{10}_{BP\mathbb{Y}}$

Then,

$\tau_{\mathbb{P}C_{BP}\mathbb{Y}}$ = {$\mathbb{P}C^{1}_{BP\mathbb{Y}}$, $\mathbb{P}C^{2}_{BP\mathbb{Y}}$, $\mathbb{P}C^{3}_{BP\mathbb{Y}}$, $\mathbb{P}C^{4}_{BP\mathbb{Y}}$, $\mathbb{P}C^{5}_{BP\mathbb{Y}}$, $\mathbb{P}C^{6}_{BP\mathbb{Y}}$, $\mathbb{P}C^{7}_{BP\mathbb{Y}}$, $\mathbb{P}C^{8}_{BP\mathbb{Y}}$, $\mathbb{P}C^{9}_{BP\mathbb{Y}}$, $\mathbb{P}C^{10}_{BP\mathbb{Y}}$, $\overline{^{ul}}C_{BP}$, $\mathbb{Y}$} is a $\mathbb{P}$- cubic bipolar Pythagorean fuzzy relative topology of $\tau^{2}_{\mathbb{P}C_{BP}}$.

3.2 Interior, Closure, Frontier and Exterior of CBPFSs

Definition 3. 23

Let ($\mathscr{M}$, $\tau_{\mathbb{P}C_{BP}\mathcal{M}}$) be a $\mathbb{P}$-CBPFT and $\mathbb{P}\mathbb{C}_{BP} \in \mathbb{P}\mathbb{C}_{BP}(\mathcal{M})$, the interior of $\mathbb{P}\mathbb{C}_{BP}$ is expressed as $\mathbb{P}C^{0}_{BP\mathbb{Y}}$ and is described as union of all $\mathbb{P}$ -cubic bipolar Pythagorean fuzzy open subsets contained in $\mathbb{P}\mathbb{C}_{BP}$. It is the greatest
$\mathbb{P}$ -cubic bipolar Pythagorean fuzzy open set contained in $\mathbb{P}\mathbb{C}_{BP}$.

Example 3.24

Consider a $\mathbb{P}$ -cubic bipolar Pythagorean fuzzy topological space as constructed in Example 3.10. Let $\mathbb{P}C_{BP_5} \in \mathbb{P}\mathbb{C}_{BP}(\mathcal{M})$ given as

$\mathbb{P}C_{BP_5}$= {[(0.73, 0.82), (0.25, 0.34)], [(-0.67, -0.43), (-0.26, -0.38)], [0.75, 0.15, -0.65, -0.45]}.

Then,

$$\left(\mathbb{P}C_{BP_5}\right)^{o} = \overline{^{ul}}C_{BP} \cup_{\mathbb{P}} \mathbb{P}C_{BP_1} = \mathbb{P}C_{BP_1}$$

Theorem 3.25

Let ($\mathscr{M}$, $\tau_{\mathbb{P}C_{BP}\mathcal{M}}$) be a $\mathbb{P}$-CBPFT and $\mathbb{P}\mathbb{C}_{BP} \in \mathbb{P}\mathbb{C}_{BP}(\mathcal{M})$. Then, $\mathbb{P}\mathbb{C}_{BP}$ is open $\mathbb{P}$-CBPFS if and only if $(\mathbb{P}\mathbb{C}_{BP})^{o} = \mathbb{P}\mathbb{C}_{BP}$.

Proof:

If $\mathbb{P}\mathbb{C}_{BP}$ is open CBPFS, then we say that the greatest open CBPFS contained in $\mathbb{P}\mathbb{C}_{BP}$ is $\mathbb{P}\mathbb{C}_{BP}$ itself. Thus,

$$(\mathbb{P}\mathbb{C}_{BP})^{o} = \mathbb{P}\mathbb{C}_{BP}.$$

Conversely, if $(\mathbb{P}\mathbb{C}_{BP})^{o} = \mathbb{P}\mathbb{C}_{BP}$, then $(\mathbb{P}\mathbb{C}_{BP})^{o}$ is open CBPFS. This implies $\mathbb{P}\mathbb{C}_{BP}$ is open CBPFS.

Theorem 3.26

Let ($\mathscr{M}$, $\tau_{\mathbb{P}C_{BP}\mathcal{M}}$) be a $\mathbb{P}$-CBPFT and $\mathbb{P}C_{BP_1}$, $\mathbb{P}C_{BP_2} \in \mathbb{P}\mathbb{C}_{BP}(\mathcal{M})$. Then,

(i) $((\mathbb{P}C_{BP})^{o})^{o} = (\mathbb{P}C_{BP})^{o}$

(ii) $\mathbb{P}C_{BP_1} \subseteq_{\mathbb{P}} \mathbb{P}C_{BP_2} \Rightarrow \left(\mathbb{P}C_{BP_1}\right)^{o} \subseteq_{\mathbb{P}} \left(\mathbb{P}C_{BP_2}\right)^{o}$

(iii) $\left(\mathbb{P}C_{BP_1} \cap_{\mathbb{P}} \mathbb{P}C_{BP_2}\right)^{o} = \left(\mathbb{P}C_{BP_1}\right)^{o} \cap_{\mathbb{P}} \left(\mathbb{P}C_{BP_2}\right)^{o}$

(iv) $\left(\mathbb{P}C_{BP_1} \cup_{\mathbb{P}} \mathbb{P}C_{BP_2}\right)^{o} \supseteq_{\mathbb{P}} \left(\mathbb{P}C_{BP_1}\right)^{o} \cup_{\mathbb{P}} \left(\mathbb{P}C_{BP_2}\right)^{o}$.

Proof: Straightforward.

Definition 3.27

Let $(\mathscr{M}, \tau_{\mathbb{P}C_{BP}\mathcal{M}})$ be a $\mathbb{P}$-CBPFT and $\mathbb{P}\mathbb{C}_{BP} \in \mathbb{P}\mathbb{C}_{BP}(\mathcal{M})$, the closure of $\mathbb{P}\mathbb{C}_{BP}$ is expressed as $\overline{\mathbb{P}\mathbb{C}_{BP}}$ and is described as the intersection of all the $\mathbb{P}$ -cubic bipolar Pythagorean fuzzy closed supersets of $\mathbb{P}\mathbb{C}_{BP}$. It is the smallest $\mathbb{P}$ -cubic bipolar Pythagorean fuzzy closed superset of $\mathbb{P}\mathbb{C}_{BP}$.

Example 3.28

Let us consider $\mathbb{P}$ -cubic bipolar Pythagorean fuzzy topological space as constructed in Example 3.10. Then, the closed CBPFSs are given as

$\left({}^{ll}C_{BP}\right)^{c}$ = {[(1, 1), (0 , 0)], [(0, 0), (-1, -1)], [0, 1, 0, -1]},

$\left({}^{lu}C_{BP}\right)^{c}$= {[(1, 1), (0, 0)], [(-1, -1), (0, 0)], [0, 1, 0, -1]},

$\left({}^{ul}C_{BP}\right)^{c}$= {[(0, 0), (1, 1)], [(0, 0), (-1, -1)], [0, 1, 0, -1]},

$\left({}^{uu}C_{BP}\right)^{c}$= {[(0, 0), (1, 1)], [(-1, -1), (0, 0)], [0, 1, 0, -1]},

$\left({}^{\overline{ll}}C_{BP}\right)^{c}$ = {[(0, 0), (1, 1)], [(-1, -1), (0, 0)], [1, 0, -1, 0]},

$\left({}^{\overline{lu}}C_{BP}\right)^{c}$ = {[(0, 0), (1, 1)], [(0, 0), (-1, -1)], [1, 0, -1, 0]},

$\left({}^{\overline{ul}}C_{BP}\right)^{c}$ = {[(1, 1), (0, 0)]], [(-1, -1), (0, 0)], [1, 0, -1, 0]},

$\left({}^{\overline{uu}}C_{BP}\right)^{c}$= {[(1, 1), (0, 0)]], [(0, 0), (-1, -1)], [1, 0, -1, 0]},

$\left(\mathbb{P}C_{BP_1}\right)^{c}$ = {[(1, 1), (0, 0)], [(-1, -1), (0, 0)], [0.79, 0.68, -0.69, -0.55]},

$\left(\mathbb{P}C_{BP_2}\right)^{c}$ = {[(0, 0), (1, 1)], [(-1, -1), (0, 0)], [0.79, 0.68, -0.69, -0.55]},

$\left(\mathbb{P}C_{BP_3}\right)^{c}$ = {[(1, 1), (0, 0)], [(0, 0), (-1, -1)], [0.79, 0.68, -0.69, -0.55]},

$\left(\mathbb{P}C_{BP_4}\right)^{c}$ = {[(0, 0), (1, 1)], [(0, 0), (-1, -1)], [0.79, 0.68, -0.69, -0.55]}.

Let $\mathbb{P}C_{BP_6} \in \mathbb{P}\mathbb{C}_{BP}(\mathcal{M})$ given as

$\mathbb{P}C_{BP_6}$= {[(0.89, 0.34), (0.64, 0.72)], [(-0.43, -0.57), (-0.22, -0.92)], [0.18, 0.72, -0.52, -0.67]}

Then,

$$\overline{\mathbb{P}C_{BP_6}} = \left({}^{\overline{ul}}C_{BP}\right)^{c} \cap_{\mathbb{P}} \left(\mathbb{P}C_{BP_1}\right)^{c} = \left(\mathbb{P}C_{BP_1}\right)^{c}$$

Theorem 3.29

Let $(\mathscr{M}, \tau_{\mathbb{P}C_{BP}\mathcal{M}})$ be a $\mathbb{P}$-CBPFT and $\mathbb{P}\mathbb{C}_{BP} \in \mathbb{P}\mathbb{C}_{BP}(\mathcal{M})$. Then, the $\overline{\mathbb{P}\mathbb{C}_{BP}}$ is closed CBPFS if and only if

$\overline{\mathbb{P}\mathbb{C}_{BP}} = \mathbb{P}\mathbb{C}_{BP}$.

Proof:

If $\mathbb{P}\mathbb{C}_{BP}$ is closed CBPFS, then we can say that the smallest closed CBPFS superset of $\mathbb{P}\mathbb{C}_{BP}$ is $\mathbb{P}\mathbb{C}_{BP}$ itself. Thus, $\overline{\mathbb{P}\mathbb{C}_{BP}} = \mathbb{P}\mathbb{C}_{BP}$.

Conversely, if $\overline{\mathbb{P}\mathbb{C}_{BP}} = \mathbb{P}\mathbb{C}_{BP}$, then $\overline{\mathbb{P}\mathbb{C}_{BP}}$ is closed CBPFS. This implies $\mathbb{P}\mathbb{C}_{BP}$ is closed CBPFS.

Definition 3.30

Let $\mathbb{P}\mathbb{C}_{BP}$ be a $\mathbb{P}$ -cubic bipolar Pythagorean fuzzy subset of $(\mathscr{M}, \tau_{\mathbb{P}C_{BP}\mathcal{M}})$, then its boundary or frontier is defined as

$$\mathrm{Fr}\,(\mathbb{P}\mathbb{C}_{\mathrm{BP}}) = \overline{\mathbb{P}\mathbb{C}_{\mathrm{BP}}} \cap_{\mathbb{P}} \overline{(\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}})}$$

Definition 3.31

Let $\mathbb{P}\mathbb{C}_{\mathrm{BP}}$ be a $\mathbb{P}$ -cubic bipolar Pythagorean fuzzy subset of $(\mathscr{M}, \tau_{\mathbb{P}\mathrm{C}_{\mathrm{BP}}\mathcal{M}})$, then the exterior is defined as

$$\mathrm{Ext}\,(\mathbb{P}\mathbb{C}_{\mathrm{BP}}) = (\overline{\mathbb{P}\mathbb{C}_{\mathrm{BP}}})^{\mathrm{c}} = (\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}})^{\mathrm{o}}$$

Example 3.32

Consider a $\mathbb{P}$ -cubic bipolar Pythagorean fuzzy topological space as constructed in Example 3.10 and $\mathbb{P}\mathrm{C}_{\mathrm{BP}_5}$ and $\mathbb{P}\mathrm{C}_{\mathrm{BP}_6}$ from Example 3.24 and 3.28. Then,

$\left(\mathbb{P}\mathrm{C}_{\mathrm{BP}_5}\right)^{\mathrm{o}} = \mathbb{P}\mathrm{C}_{\mathrm{BP}_1}$, $\overline{\mathbb{P}\mathrm{C}_{\mathrm{BP}_5}} = \left({}^{\overline{\mathrm{ul}}}\mathrm{C}_{\mathrm{BP}}\right)^{\mathrm{c}}$, $\mathrm{Fr}\,(\mathbb{P}\mathrm{C}_{\mathrm{BP}_5}) = {}^{\overline{\mathrm{ul}}}\mathrm{C}_{\mathrm{BP}}$, $\mathrm{Ext}\,(\mathbb{P}\mathrm{C}_{\mathrm{BP}_5}) = {}^{\overline{\mathrm{ul}}}\mathrm{C}_{\mathrm{BP}}$.

$\left(\mathbb{P}\mathrm{C}_{\mathrm{BP}_6}\right)^{\mathrm{o}} = \mathbb{P}\mathrm{C}_{\mathrm{BP}_1}$, $\overline{\mathbb{P}\mathrm{C}_{\mathrm{BP}_6}} = \left(\mathbb{P}\mathrm{C}_{\mathrm{BP}_1}\right)^{\mathrm{c}}$, $\mathrm{Fr}\,(\mathbb{P}\mathrm{C}_{\mathrm{BP}_6}) = \mathbb{P}\mathrm{C}_{\mathrm{BP}_1}$, $\mathrm{Ext}\,(\mathbb{P}\mathrm{C}_{\mathrm{BP}_6}) = \mathbb{P}\mathrm{C}_{\mathrm{BP}_1}$.

Theorem 3.33

Let $(\mathscr{M}, \tau_{\mathbb{P}\mathrm{C}_{\mathrm{BP}}\mathcal{M}})$ be a $\mathbb{P}$-CBPFT and $\mathbb{P}\mathbb{C}_{\mathrm{BP}} \in \mathbb{P}\mathbb{C}_{\mathrm{BP}}(\mathcal{M})$. Then

(1) $(\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{o}})^{\mathrm{c}} = \overline{(\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}})}$

(2) $(\overline{\mathbb{P}\mathbb{C}_{\mathrm{BP}}})^{\mathrm{c}} = (\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}})^{\mathrm{o}}$

(3) $\mathrm{Ext}\,(\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}}) = \mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{o}}$

(4) $\mathrm{Ext}\,(\mathbb{P}\mathbb{C}_{\mathrm{BP}}) = (\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}})^{\mathrm{o}}$

(5) $\mathrm{Fr}\,(\mathbb{P}\mathbb{C}_{\mathrm{BP}}) = \mathrm{Fr}\,(\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}})$

Proof:

(1) The proof is obvious.

(2) The proof is obvious.

(3) $\mathrm{Ext}\,(\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}}) = \overline{(\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}})^{\mathrm{c}}}$

$\mathrm{Ext}\,(\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}}) = ((\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{o}}$

$\mathrm{Ext}\,(\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}}) = \mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{o}}$

(4) $\mathrm{Ext}\,(\mathbb{P}\mathbb{C}_{\mathrm{BP}}) = (\overline{\mathbb{P}\mathbb{C}_{\mathrm{BP}}})^{\mathrm{c}}$

$\mathrm{Ext}\,(\mathbb{P}\mathbb{C}_{\mathrm{BP}}) = (\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}})^{\mathrm{o}}$

(5) $\mathrm{Fr}\,(\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}}) = \overline{(\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}})} \cap_{\mathbb{P}} \overline{((\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}})^{\mathrm{c}})}$

$= \overline{(\mathbb{P}\mathrm{C}_{\mathrm{BP}}^{\mathrm{c}})} \cap_{\mathbb{P}} \overline{\mathbb{P}\mathbb{C}_{\mathrm{BP}}}$

$= \mathrm{Fr}\,(\mathbb{P}\mathbb{C}_{\mathrm{BP}})$.

3.3 $\mathbb{P}$- Cubic Bipolar Pythagorean Fuzzy Basis

Definition 3.34

Let $(\mathscr{M}, \tau_{\mathbb{P}\mathrm{C}_{\mathrm{BP}}\mathcal{M}})$ be a $\mathbb{P}$-CBPFT. Then $\mathbb{B} \subseteq \tau_{\mathbb{P}\mathrm{C}_{\mathrm{BP}}\mathcal{M}}$, is called $\mathbb{P}$- Cubic Bipolar Pythagorean Fuzzy Basis for $\tau_{\mathbb{P}\mathrm{C}_{\mathrm{BP}}\mathcal{M}}$ if for every $\mathbb{P}\mathbb{C}_{\mathrm{BP}} \in \tau_{\mathbb{P}\mathrm{C}_{\mathrm{BP}}\mathcal{M}}$, $\exists\ \mathfrak{B} \in \mathbb{B}$ such that

$$\mathbb{P}\mathbb{C}_{\mathrm{BP}} = \cup_{\mathbb{P}}\ \mathfrak{B}.$$

IV. Advanced MCDM Framework: CRITIC-MAIRCA Integration under CBPFS

To operationalize this highly abstract mathematical topology into actionable clinical intelligence, an advanced Multi-Criteria Decision-Making (MCDM) model is constructed. The original manuscript theoretically suggests combining the CRITIC and MAIRCA algorithms within the CBPFS domain.

The CRITIC (Criteria Importance Through Intercriteria Correlation) method is a revolutionary objective weighting algorithm. It completely eliminates the need for subjective human weighing by calculating criterion weights entirely based on internal data contrast (Standard Deviation) and systemic conflict (Correlation). MAIRCA (Mult Attributive Ideal-Real

Comparative Analysis) evaluates decision alternatives by computing the total geometric gap between an empirically observed state and a theoretically perfect (Ideal) prior state.

The algorithmic integration of these two methods under the CBPFS topological framework provides an incredibly robust, self-correcting diagnostic engine.

4.1.1 Algorithm for CBPF CRITIC Method

1. Input the CBPF matrix D according to positive and negative feedback ratio provided by the decision maker.

$$D_{m\times n} = (d_{ij})_{m\times n} = (\mu_A^+, \vartheta_A^+, \mu_A^-, \vartheta_A^-)_{m\times n}$$

2. Compute the score values of $S(a_{ij})$ where $i = 1,2,3 \ldots m$ and $j = 1,2,3 \ldots . n$ using the equation

$$S(A_{m\times n}) = S(a_{ij}) = 1 - \left|\frac{((\mu_A^+)^2 - (\vartheta_A^+)^2 + (\mu_A^-)^2 - (\vartheta_A^-)^2))}{2}\right|.$$

3. Utilize CRITIC method to compute the attributes weight.

3(i) The decision matrix is normalized using the following equation

$$\text{NDM} = [d_{ij}{}^*]_{m\times n} = \frac{d_{ij} - d_j^{worst}}{d - d_j^{worst}} \text{ where } d_j^{worst} = \min_i d_{ij} \text{ and } d_j^{best} = \max_i d_{ij}$$

3(ii) Compute the correlation coefficient of the attributes A_i to A_n using the following equation

$$r_{ik} = \frac{\sum_{i=1}^{m}(d_{ij}{}^* - \overline{d_j{}^*})(d_{ik}{}^* - \overline{d_k{}^*})}{\sqrt{\sum_{i=1}^{m}(d_{ij}{}^* - \overline{d_j{}^*})^2\ (d_{ik}{}^* - \overline{d_k{}^*})^2}} \text{ where } \overline{d_j{}^*} = \frac{1}{m}\sum_{i=1}^{m} d_{ij}{}^* \text{ and } \overline{d_k{}^*} = \frac{1}{m}\sum_{i=1}^{m} d_{ik}{}^*$$

3(iii) Compute the Standard deviation for each attribute

$$s_i = \sqrt{\frac{1}{m}\sum_{i=1}^{m}(d_{ij}{}^* - \bar{d}^*)^2}, \text{ for every j=1,2,3…n.}$$

3(iv) Calculate the deviation degree φ of criterion C_j from the other criteria

$$\varphi_j = s_i \sum_{k=1}^{n}(1 - r_{ik}), \text{ j=1,2,3…. n}$$

3(v) Estimate the weights of the attributes $w_j = \frac{\varphi_j}{\sum_{j=1}^{n}\varphi_j}$ for every j=1,2,3……. N

4.1.2 Algorithm for CBPF MAIRCA Method

1. Input the CBPF matrix S according to positive and negative feedback ratio provided by the decision maker.

$$D_{m\times n} = (d_{ij})_{m\times n} = (\mu_A^+, \vartheta_A^+, \mu_A^-, \vartheta_A^-)_{m\times n}$$

2. Compute the score values of $S(d_{ij})$ where $i = 1,2,3 \ldots m$ and $j = 1,2,3 \ldots . n$ using the equation

$$S(A_{m\times n}) = S(d_{ij}) = 1 - \left|\frac{((\mu_A^+)^2 - (\vartheta_A^+)^2 + (\mu_A^-)^2 - (\vartheta_A^-)^2))}{2}\right|.$$

3. Utilize CRITIC method to compute the attributes weight.

4. Evaluate the priority for an indicator. When the decision maker is neutral, the indicators is the same. Then the priority for all criteria is the same and is calculated using the following relation:

$$\wp_{H_j} = \frac{1}{m}, j = 1,2,3, \ldots \ldots, n$$

5. Compute the quantities $g_{pij} = \wp_{H_j}. w_j, i = 1,2,3 \ldots \ldots m; j = 1,2,3 \ldots \ldots n$

6. Compute the quantities g_{rij} according to the following equations:

$$g_{rij} = g_{pij} \cdot \left(\frac{d_{ij} - d_i{}^{worst}}{d_i{}^{Best} - d_i{}^{worst}}\right) \text{ if } j \in \text{Benefit criterion}$$

$$g_{rij} = g_{pij} \cdot \left(\frac{d_{ij} - d_i{}^{Best}}{d_i{}^{worst} - d^{Best}}\right) \text{ if } j \in \text{non} - \text{Benefit criterion}$$

7. Compute ρ_{ij} according to the following equation: $\rho_{ij} = g_{pij} - g_{rij}$.

8. Summation of ρ_j values using the following equation: $\theta_i = \sum_{i=1}^{m} \rho_{ij}$.
Rank the alternatives by using the principle that the alternative with smallest θ_i is the best followed by the other alternatives.

4.2 Numerical example

Extensive Clinical Case Study: Chronic Cancer Diagnosis

The field of oncology entails several decisions to be made by healthcare professionals. Such factors like tumour stage, tumour grade, biomarker value, patient condition, therapy response, and associated risks might be unknown, vague, and even inconsistent. Thus, the choice of the most suitable treatment regime depends on the consideration of various clinical parameters.

In order to illustrate the applicability of the introduced theoretical model, the proposed algorithm will be implemented into the oncology case study where different treatment protocols are assessed by the multidisciplinary tumor board for the patients diagnosed with advanced chronic cancer. The introduced CBPF approach will be used to depict the uncertain clinical data and assess the criteria.

Decision Parameters Contextualization

Alternatives (Treatment Protocols): The panel evaluates four distinct protocols ($m = 4$):

- A_1: Standard Systemic Chemotherapy (Broad-spectrum cell destruction)
- A_2: Targeted Radiation Therapy (Localized intervention)
- A_3: Advanced Immunotherapy (e.g., Checkpoint inhibitors or CAR-T cell therapy)
- A_4: Combined Modality Palliative Care (Focus on symptom management)

Criteria (Clinical Indicators): The protocols are evaluated against four independent clinical metrics ($n = 4$):

- C_1: Efficacy and Tumor Regression Rate (Benefit Criterion)
- C_2: Systemic Toxicity and Adverse Organ Events (Cost/Non-Benefit Criterion)
- C_3: Long-term Remission Probability (Benefit Criterion)
- C_4: Treatment Financial Burden and Accessibility (Cost/Non-Benefit Criterion)

Matrix Formulation and Score Aggregation

The expert panel assesses the alternatives. Due to the severe ambiguity in oncology (e.g., overlapping confidence intervals in clinical trials), CBPFS structures elegantly capture both the interval of hesitancy and the exact diagnostic metrics. Following the initial input matrix generation, the defuzzification process detailed in Step 2 is applied. The continuous score matrix S is generated, translating the multi-dimensional fuzzy sets into actionable scalar values:

STEP 1: Input the CBPF interval valued matrix D according to positive and negative feedback ratio provided by the decision maker.

	C1 (Efficacy)	**C2** (Toxicity)
A1	$\langle\{[-0.6, -0.4], [-0.5, -0.3], [0.6, 0.5], [0.1, 0.4]\}, \{-0.6, -0.7, 0.8, 0.5\}\rangle$	$\langle\{[-0.7, -0.4], [-0.3, -0.2], [0.2, 0.5], [0.3, 0.4]\}, \{-0.7, -0.5, 0.6, 0.8\}\rangle$

A2	$\langle\{[-0.7,\ -0.5],\ [-0.5,\ -0.3],\ [0.3,\ 0.7],\ [0.5,\ 0.7]\},\ \{-0.3, -0.3, 0.8, 0.4\}\rangle$	$\langle\{[-0.8,\ -0.6],\ [-0.5,\ -0.1],\ [0.5,\ 0.6],\ [0.2,\ 0.4]\},\ \{-0.6, -0.7, 0.8, 0.5\}\rangle$
A3	$\langle\{[-0.5,\ -0.4],\ [-0.6,\ -0.5],\ [0.4,\ 0.5],\ [0.4,\ 0.6]\},\ \{-0.2, -0.7, 0.4, 0.2\}\rangle$	$\langle\{[-0.7,\ -0.6],\ [-0.3,\ -0.1],\ [0.1,\ 0.3],\ [0.1,\ 0.4]\},\ \{-0.4, -0.8, 0.3, 0.5\}\rangle$
A4	$\langle\{[-0.6,\ -0.4],\ [-0.9,\ -0.3],\ [0.4,\ 0.7],\ [0.1,\ 0.4]\},\ \{-0.6, -0.9, 0.8, 0.5\}\rangle$	$\langle\{[-0.4,\ -0.4],\ [-0.7,\ -0.2],\ [0.5,\ 0.6],\ [0.3,\ 0.6]\},\ \{-0.7, -0.5, 0.7, 0.4\}\rangle$

	C3 (Remission)	**C4** (Financial Burden)
A1	$\langle\{[-0.6,\ -0.5],\ [-0.7,\ -0.5],\ [0.3,\ 0.4],\ [0.7,\ 0.3]\},\ \{-0.5, -0.4, 0.7, 0.5\}\rangle$	$\langle\{[-0.5,\ -0.3],\ [-0.7,\ -0.5],\ [0.4,\ 0.7],\ [0.7,\ 0.3]\},\ \{-0.3, -0.8, 0.7, 0.4\}\rangle$
A2	$\langle\{[-0.5,\ -0.3],[-0.4,\ -0.3],\ [0.6,\ 0.8],\ [0.3,\ 0.7]\},\ \{-0.4, -0.7, 0.8, 0.5\}\rangle$	$\langle\{[-0.6,\ -0.4],\ [-0.8,\ -0.7],\ [0.5,\ 0.9],\ [0.7,\ 0.3]\},\ \{-0.5, -0.4, 0.7, 0.5\}\rangle$
A3	$\langle\{[-0.7,\ -0.4],\ [-0.6,\ -0.2],\ [0.4,\ 0.5],\ [0.3,\ 0.6]\},\ \{-0.5, -0.7, 0.8, 0.6\}\rangle$	$\langle\{[-0.6,\ -0.2],\ [-0.4,\ -0.3],\ [0.0,\ 0.2],\ [0.5,\ 0.4]\},\ \{-0.1, -0.9, 0.4, 0.4\}\rangle$
A4	$\langle\{[-0.6,\ -0.5],\ [-0.9,\ -0.3],\ [0.1,\ 0.4],\ [0.2,\ 0.4]\},\ \{-0.6, -0.9, 0.8, 0.1\}\rangle$	$\langle\{[-0.7,\ -0.3],\ [-0.9,\ -0.7],\ [0.7,\ 0.8],\ [0.5,\ 0.2]\},\ \{-0.2, -0.4, 0.5, 0.6\}\rangle$

STEP 2: Compute the score values of $S(d_{ij})$ where $i = 1,2,3 \ldots m\ and\ j = 1,2,3 \ldots .n$ using the equation

$$S(D_{m\times n}) = S(d_{ij}) = \left|\frac{((\mu_A^+)^2 - (\vartheta_A^+)^2 + (\mu_A^-)^2 - (\vartheta_A^-)^2))}{2}\right|$$

Score value	**C1**	**C2**	**C3**	**C4**
A1 (Chemotherapy)	-0.0333	-0.1400	0.3433	0.3100
A2 (Radiation)	0.3700	0.1100	0.21	0.4767
A3(Immunotherapy)	0.4067	0.0267	0.1067	-0.0133
A4 (Palliative)	0.0233	0.2833	0.2733	0.4667

STEP 3: Utilize CRITIC method to compute the attributes weight.
Step 3(i): The decision matrix is normalized using the following equation

NDM	**C1**	**C2**	**C3**	**C4**
A1	0.0424	0.0000	0.0000	-0.6599
A2	0.4458	0.5906	0.5634	-1.0000
A3	0.4824	0.3937	1.0000	0.0000
A4	0.0991	1.0000	0.2958	-0.9796

Step 3(ii): Compute the correlation coefficient of the attributes A_1 to A_4

rjk	**A1**	A2	A3	A4
A1	1.0000	0.0757	0.9118	0.4216
A2	0.0757	1.0000	0.1885	-0.4477
A3	0.9118	0.1885	1.0000	0.6287
A4	0.4216	-0.4477	0.6287	1.0000

Step 3(iii): Compute the Standard deviation for each attribute

Attribute	**A1**	A2	A3	A4
SD	0.2288	0.4161	0.4246	0.4667

Step 3(iv): Calculate the deviation degree φ of criterion C_j from the other criteria

1-rjk	C1	C2	C3	C4
C1	0.0000	0.9243	0.0882	0.5784
C2	0.9243	0.0000	0.8115	1.4477
C3	0.0882	0.8115	0.0000	0.3713
C4	0.5784	1.4477	0.3713	0.0000

Quantity of information	
A1	0.5927
A2	1.7408
A3	0.9642
A4	1.5855
Total	4.8830

Step 3(v): Estimate the weights of the attributes.

Weight	A1	A2	A3	A4
w_j	0.1214	0.3565	0.1975	0.3247

CBPF CRITIC MAIRCA:

STEP1, STEP2, STEP3 as in CRITIC method

STEP 4: Evaluate the priority for an indicator. When the decision maker is neutral, the indicators is the same. Then the priority for all criteria is the same and is calculated using the following relation:

$$\wp_{H_j} = \frac{1}{m}, j = 1,2,3, \dots \dots, n$$

STEP 5: Compute the quantities $\mathscr{g}_{pij} = \wp_{H_j} . w_j, i = 1,2,3 \dots \dots m; j = 1,2,3 \dots \dots n$

$\mathscr{g}_{pij}$	C1	C2	C3	C4
A1	0.0272	0.0989	0.0403	0.0836

A2	0.0272	0.0989	0.0403	0.0836
A3	0.0272	0.0989	0.0403	0.0836
A4	0.0272	0.0989	0.0403	0.0836

STEP 6: Compute the quantities $\mathcal{g}_{\mathrm{rij}}$

$\mathcal{g}_{\mathrm{rij}}$	C1	C2	C3	C4
A1	0.0012	0.0000	0.0000	-0.0551
A2	0.0121	0.0584	0.0227	-0.0836
A3	0.0131	0.0390	0.0403	0.0000
A4	0.0027	0.0989	0.0119	-0.0819

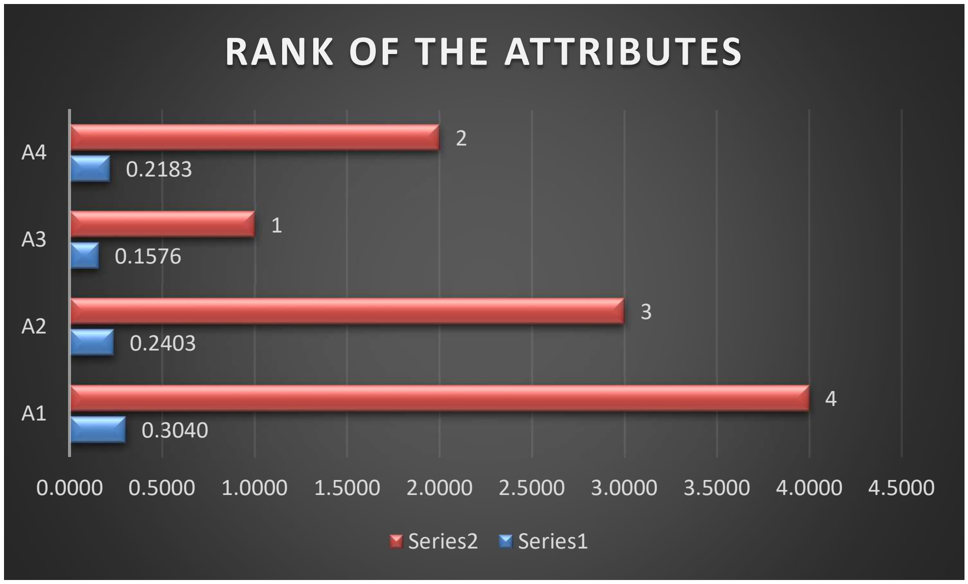
STEP 7: Compute ρ_{ij} according to the following equation: $\rho_{\mathrm{ij}} = \mathcal{g}_{\mathrm{pij}} - \mathcal{g}_{\mathrm{rij}}$.

$\mathcal{g}_{\mathrm{pij}} - \mathcal{g}_{\mathrm{rij}}$	**C1**	**C2**	**C3**	**C4**
A1	**0.0260**	**0.0989**	**0.0403**	**0.1387**
A2	**0.0151**	**0.0405**	**0.0176**	**0.1671**
A3	**0.0141**	**0.0600**	**0.0000**	**0.0836**
A4	**0.0245**	**0.0000**	**0.0284**	**0.1654**

STEP 8: Rank the alternatives by using the principle that the alternative with smallest θ_{i} is the best followed by the other alternatives.

	θ_{i}	Rank
A1	0.3040	4
A2	0.2403	3
A3	0.1576	1
A4	0.2183	2

4.3 Result and discussion



The acquired results reveal that the CBPF topological decision-making model performs efficiently in dealing with uncertainties and ambiguities in the clinical data set. The acquired score values for four treatment alternatives A1, A2, A3, and A4 are 0.3040, 0.2403, 0.1576, and 0.2183, respectively. From the ranking method, the ranking order of the treatment alternatives is given as: A3>A4>A2>A1.

Hence, Advanced Immunotherapy (A3) turns out to be the best treatment alternative followed by Combined Modality Palliative Care (A4), Targeted Radiation Therapy (A2), and Standard Systemic Chemotherapy (A1).

Moreover, the CBPF topological method handles small deviations in the clinical parameters. For instance, the small deviation in the value of laboratory measurement parameters denoted by the variation in C2 is within the topological neighborhood of the corresponding parameter. Hence, the ranking A3>A4>A2>A1 remains intact under small perturbation of the clinical parameters.

4.4 Conclusion

It is found through the numerical illustration that the proposed CBPF topology-based decision-making model is capable of solving uncertain and vague clinical data problems efficiently. Among the four treatment methods taken into consideration, Advanced Immunotherapy (A3) comes first with the highest preference, followed by Combined Modality Palliative Care (A4), Targeted Radiation Therapy (A2), and Standard Systemic Chemotherapy (A1).

The persistence of this ordering in the presence of slight variations in input data confirms the robustness of the proposed CBPF topology-based methodology for clinical decision-making problems.

4.5 Future work

For future research, one could concentrate on the development of more extensive analysis of stability and sensitivity in CBPF topological decision-making model. For instance, the framework could be extended to examine the influence of greater perturbations, varied forms of uncertainty in clinical data, and actual medical data sets. Other areas of study could involve exploring the inclusion of CBPF topological structures with machine learning, image processing, and other multi-criteria decision-making methodologies.

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