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LIPSCHITZ LABELING OF SOME STAR GRAPHS

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Abstract

Lipschitz labelling on graph is a function f from the vertex set and edge set of the graph to $\{1,2,\ldots,m+n\}$ where $m=|V(G)|$ and $n=|E(G)|$ such that

(i) f is bijective.
(ii) $f(v)\in\{1,2,\ldots,m\}$ for all $v\in V(G)$
(iii) $f(x)\in\{m+1,m+2,\ldots,m+n\}$ for all $x\in E(G)$
(iv) There exists a positive integer L such that

$$|f(x)-f(y)|\le L\left|x_f-y_f\right|$$

for all $x,y\in E(G)$ and $x_f=\min\{f(u),f(v)\}$ for $x=uv$.

We study the problem of Lipschitz labelling within the context of structured graphs. Specifically, we establish the Lipschitz constants for three families of star-derived graphs: twinkling star graphs with ray of length $2n\ (n\ge 1)$, triangular graphs $\Lambda_{n,}$, and bistar graphs $B_{n,m}$.

Keywords: Twinkling star graph, triangular graph, bistar graph.

1 Introduction

Graph labeling is a fundamental and richly explored area of graph theory, concerned with the assignment of integers-typically to vertices, edges, or both subject to specific mathematical constraints. Since its seminal introduction by Rosa in the 1960s [6], the field has proliferated into a vast family of labeling schemes, each with its own unique rules and applications, ranging from radar and coding theory to circuit design.

Following the foundational work on graph labelings, the field diversified through new categories such as Sedlacek's magic labelings [7] and Cahit's more flexible cordial labelings [3]. Research in this area has since proliferated, encompassing specialized studies like those on prime cordial labelings [5] and is comprehensively tracked in Gallian's dynamic survey [4]..

A recent and intriguing addition to this landscape is the concept of Lipschitz labeling, introduced by Umapathi and Senthil Amutha [9]. This labeling paradigm draws its inspiration from the classical notion of Lipschitz continuity in mathematical analysis. In the discrete context of graphs, a Lipschitz labeling requires that the difference between the labels of any two adjacent vertices be bounded by a constant, effectively ensuring that the labeling is a "Lipschitz function" on the graph's vertex set. The central problem is to determine the smallest such constant, known as the Lipschitz constant, for which a proper labeling exists. The initial investigation [9] established foundational results, proving that star graphs and comb graphs admit such labelings, while cycle and cyclic graphs do not. This dichotomy highlights the dependency of the Lipschitz labeling property on the underlying graph structure and opens the door for a systematic classification of graph families.

Building directly upon this work, our paper delves into the Lipschitz labeling problem for several intricate graph structures derived from stars. Star graphs, due to their simple yet highly connected central vertex, often serve as critical test cases in labeling problems. We extend the analysis to more complex, star-like architectures:

- The Twinkling Star Graph with rays of length $2n(n \geq 1)$,
- The Triangular Graph Λ_n, and
- The Bistar Graph $B_{n,m}$.

For each of these families, we investigate the existence of a Lipschitz labeling and determine their precise Lipschitz constants, thereby contributing to a more comprehensive understanding of this new labeling scheme's behavior.
The remainder of this paper is organized as follows. Section 2 provides the necessary preliminary definitions and formalizes the concept of Lipschitz labeling. Section 3 presents our main analytical results, detailing the Lipschitz labeling constructions and proofs for the aforementioned graph families. Finally, Section 4 concludes the paper by summarizing our findings and suggesting potential directions for future research.

2 Basic definitions

This section outlines the preliminary concepts and definitions that form the basis of our investigation. We begin by formalizing key graph-theoretic notions, with an emphasis on labeling schemes. The core framework for our analysis is provided by the concept of Lipschitz labeling, introduced by Umapathi and Senthil Amutha [9] as a development upon classical irregular labelings. We present the formal definition below:

Definition 1 [1] A function $f(t, y)$ is said to satisfy a Lipschitz condition in the variable y on a set D in $\mathbb{R}$ if there exists a constant $L > 0$ such that

$$| f(t, y_1) - f(t, y_2| \leq L|y_1 - y_2 |$$

whenever both points (t, y_1) and (t, y_2) are in D. The constant L is called a Lipschitz constant for f.

Definition 2 [9] Let $G=(V(G),E(G))$ be a finite simple graph with $|V(G)|=m$ and $|E(G)|=n\geq 1$. The function $f\colon V(G)\cup E(G)\to\{1,2,\dots,m+n\}$ is said to be Lipschitz labeling if f satisfy the following conditions.

(i) f is bijective.
(ii) $f(v)\in\{1,2,\dots,m\}$ for all $v\in V(G)$
(iii) $f(x)\in\{m+1,m+2,\dots,m+n\}$ for all $x\in E(G)$
(iv) There exists a positive integer L such that

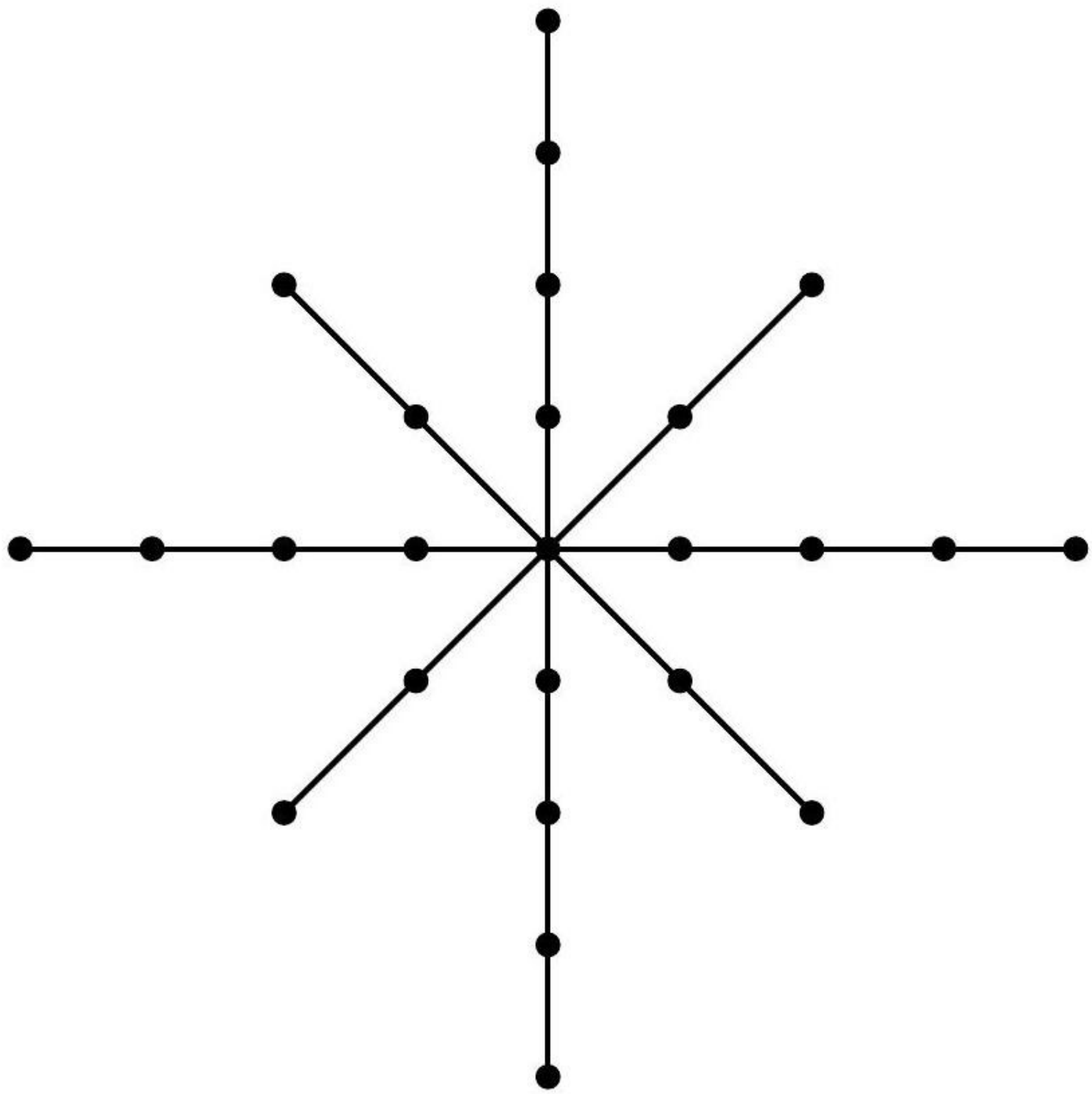
$$|f(x)-f(y)|\leq L\left|x_f-y_f\right|$$

for all $x,y\in E(G)$ and $x_f=\min\{f(u),f(v)\}$ for $x=uv$.

Remark 1 In the above definition, we assume that x and y are need not be distinct.

Definition 3 A twinkling star with ray of length $2n, n\geq 1$, is a tree and is obtained by adjoing P_{2n-1} with half of the pendant vertices of $K_{1,8}$ and adjoing P_{2n-3} with remaining pendant vertices of $K_{1,8}$. Here, if $P_{\text{-ve}}$, then there is no adjoing.

Example 1 The twinkling star graph with ray of length 4 is given by



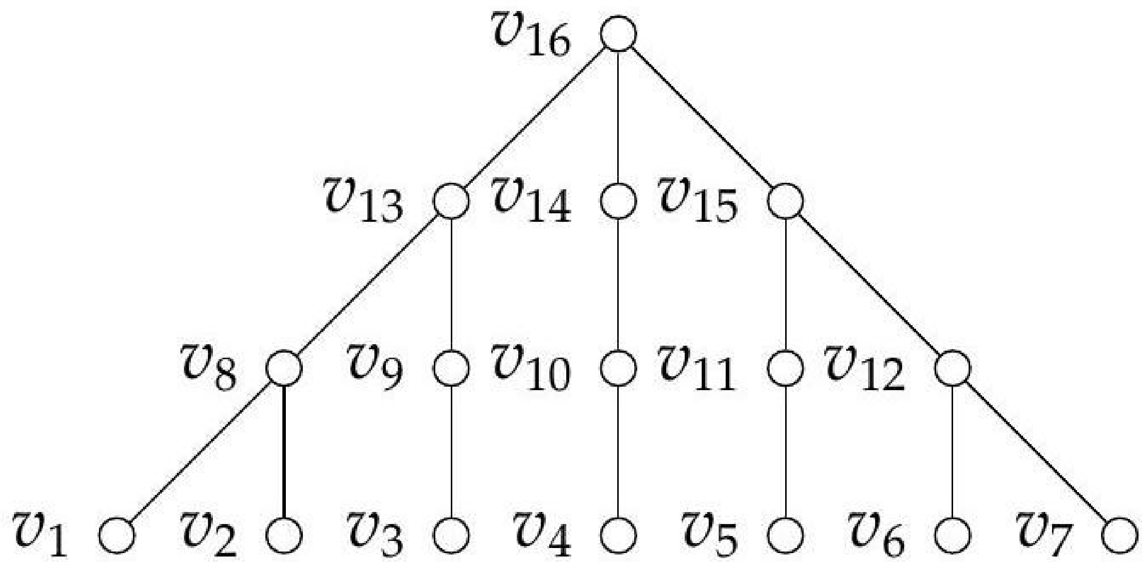
Definition 4 Let $h \geq 1$ and $v_1, v_2, \dots, v_{h-1}, v_h, v_{h+1}, \dots, v_{2h+1}$ be the vertices of P_{2h}. A triangular tree graph or triangular graph with height h is donoted by Λ_h and is obtained from P_{2h} by the following procedure.

- Adjoining P_1 with v_2 and v_{2h}
- Adjoining P_2 with v_3 and v_{2h-1}
- Adjoining P_3 with v_4 and v_{2h-2}
- ...
- Adjoining P_{h-1} with v_{h-1} and v_{h+1}
- Adjoining P_h with v_h

Remark 2 - The vertex v_h is called the apex of Λ_h.

- A triangular graph with height h has $h+1$ levels. The pendant vertices are called the level 1 vertices, the vertices directly above the level 1 vertices are called level 2 vertices, and so on.
- The apex is the vertex in level $h+1$.

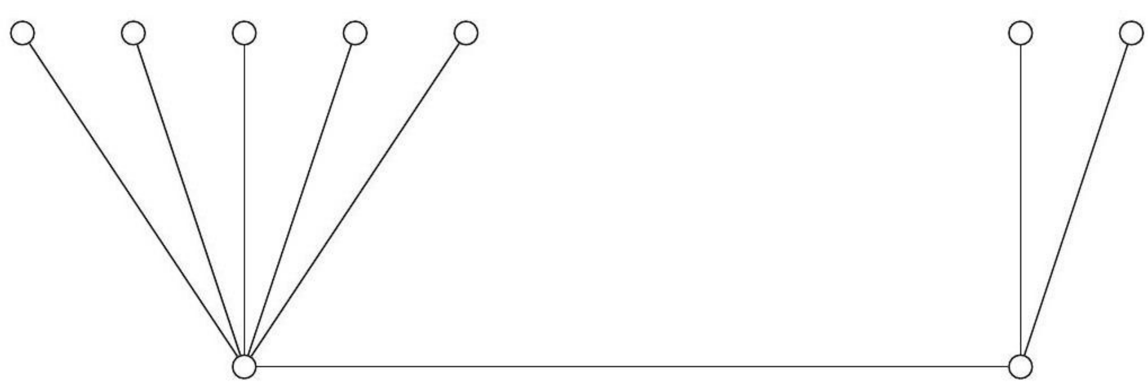
Example 2 A triangular graph Λ_3 is given by



Here, $v_1, \dots, v_7$ are level 1 vertices, $v_8, \dots, v_{12}$ are level 2 vertices, v_{13}, v_{14}, v_{15} are level 3 vertices, and v_{16} is the level 4 vertex.

Definition 5 A Bistar graph is the graph obtained by joining the centre (apex) vertices of $K_{1,n}$ and $K_{1,m}$ by an edge and it is denoted by $B_{n,m}$

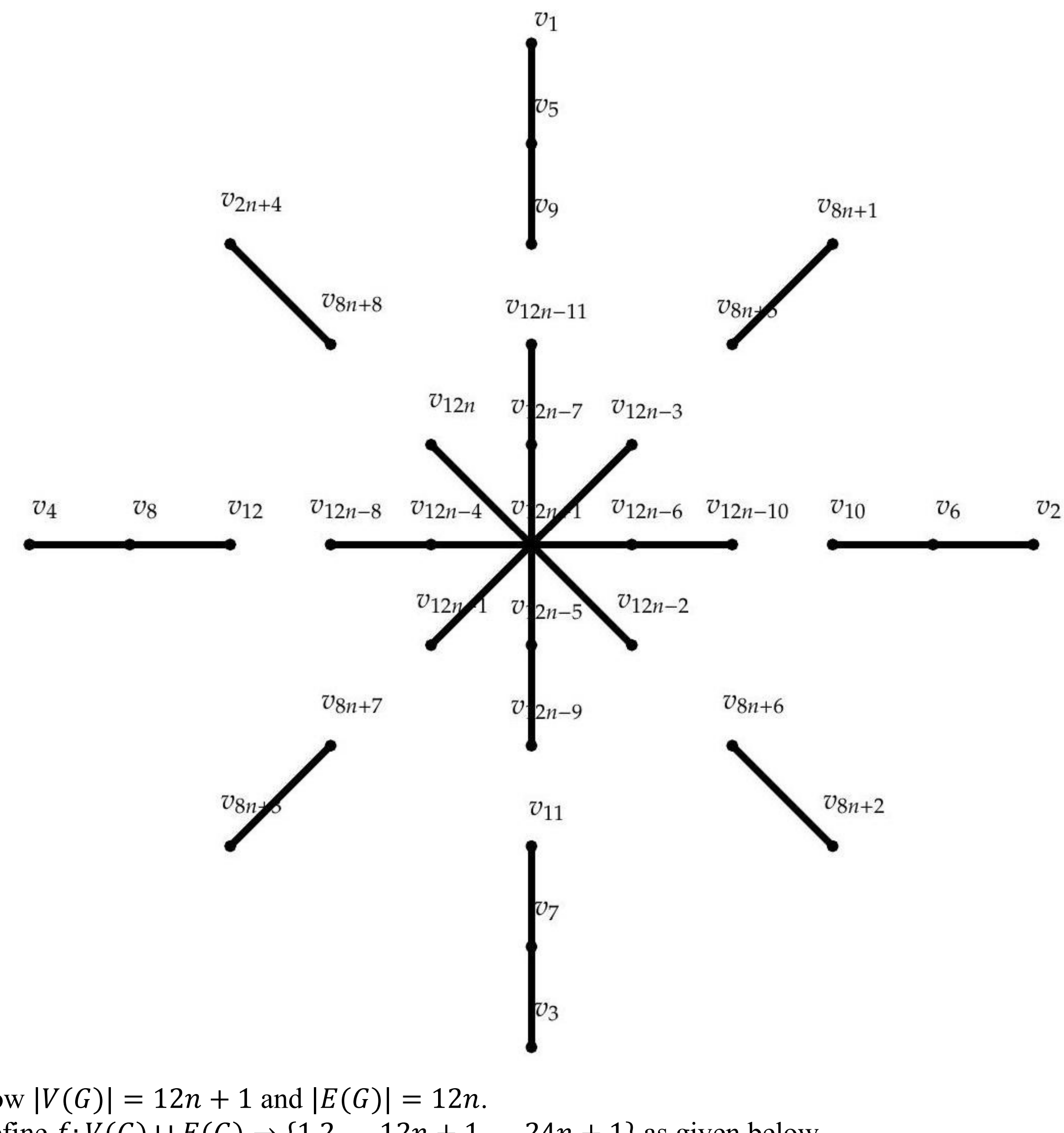
Example 3 The $B_{5,2}$ is given by



3 Main results

This section presents the core contributions of this work: an investigation into the existence and explicit construction of Lipschitz labelings for specific families of star graphs. The results are substantiated by rigorous theoretical proofs, augmented with illustrative examples to demonstrate the application of the labeling schemes.

Theorem 1 A twinkling star graph is a Lipschitz graph.
Proof. Let G be a twinkling star graph with ray of length of $2n, n \geq 1$. Consider G as given below.



Now $|V(G)| = 12n + 1$ and $|E(G)| = 12n$.
Define $f: V(G) \cup E(G) \rightarrow \{1,2, \ldots ,12n + 1, \ldots ,24n + 1\}$ as given below.

$$f(v_i) = i$$

and

$$f(v_iv_j) = 12n + 1 + \min\{i, j\}$$

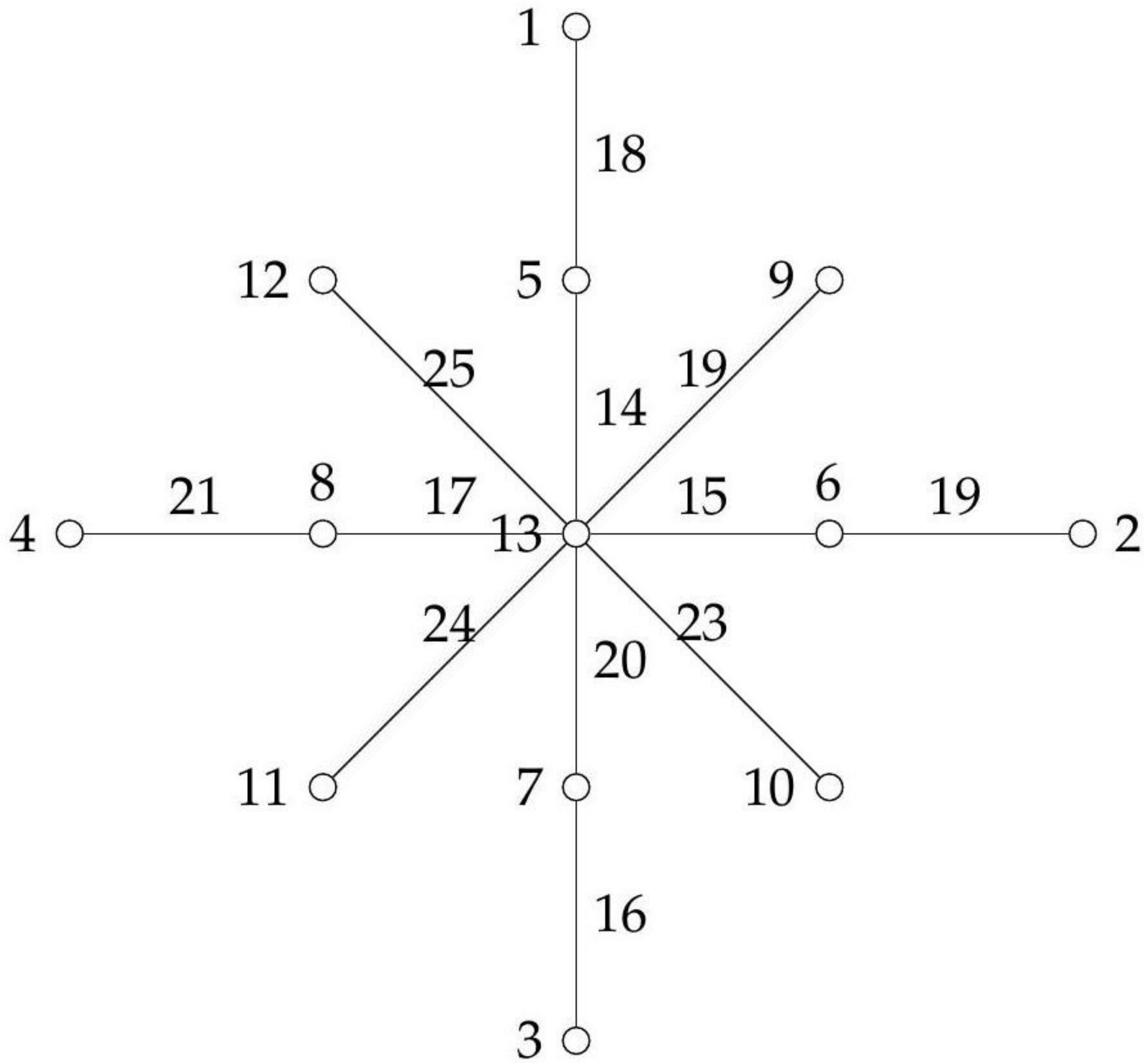
Let $x = v_iv_j$ and $y = v_kv_l$ be any two egdes in the graph.
Then

$$\begin{aligned}|f(x) - f(y)| &= |(12n + 1 + \min\{i, j\}) - (12n + 1 + \min\{k, l\})| \\ &= |\min\{i, j\} - \min\{k, l\}| \\ &= \left|\min\{f(v_i), f(v_j)\} - \min\{f(v_k), f(v_l)\}\right| \text{ since } f(v_n) = n \\ &= \left|x_f - y_f\right|\end{aligned}$$

From the above two discussion, we have

$$|f(x) - f(y)| \leq L \cdot \left|x_f - y_f\right|$$

for all $x, y \in E(G)$ with $L = 1$.
Hence G is a Lipschitz graph.
Example 4 A lipschitz labeling of the twinkling star graph with ray of length 2 is given by



Theorem 2 A triangular tree $\Lambda_h, h \geq 1$, is a Lipschitz graph.
Proof. Let G be a triangular tree graph $\Lambda_h, h \geq 1$.
Then $|V(G)| = (h+1)^2$ and $|E(G)| = h(h+2)$.
The graph has $h+1$ levels.
Let $f: V(G) \cup E(G) \to \{1,2,3, \ldots, (h+1)^2, \ldots,\}$ with the following condtions.

Vertex labeling

- The bottom level (level 1) contains $2h+1$ vertices. Each vertex in the bottom level is assigned a unique label $1,2, \ldots, 2h+1$, sequentially from left to right.
- Let v be a vertex on the bottom level with label i, where $2 \leq i \leq 2h$. The vertex directly above v is assigned the label $i+2h$. As a result, the second level vertices are labeled sequentially from $2h+1$ to $4h$, corresponding to their positions directly above the bottom-level vertices.

- For a vertex u at level 2 , with label i satisfying $2h+3 \leq i \leq 4h-1$, the vertex directly above u is assigned the label $i+2h-2$.
- Repeat this process iteratively for each subsequent level.
- The labeling ensures that the function $f(v)$, which maps each vertex v to its label, covers the range $\{1,2,3,\ldots,(h+1)^2\}$, providing a unique label to each vertex.

Edge lebeling

$$f(v_i v_j) = (h+1)^2 + \min\{f(v_i), f(v_j)\}$$

for all egde $v_i v_j \in E(G)$.
Let $x = v_i v_j$ and $y = v_k v_l$ be any two egdes in the graph.
Then

$$\begin{aligned}|f(x) - f(y)| &= \left|\left((h+1)^2 + \min\{f(v_i), f(v_j)\}\right) - ((h+1)^2 + \min\{f(v_k), f(v_l)\})\right| \\ &= \left|\min\{f(v_i), f(v_j)\} - \min\{f(v_k), f(v_l)\}\right| \\ &= \left|x_f - y_f\right|\end{aligned}$$

From the above two discussion, we have

$$|f(x) - f(y)| \leq L \cdot \left|x_f - y_f\right|$$

for all $x, y \in E(G)$ with $L = 1$.
Hence Λ_h is a Lipschitz graph.
Theorem 3 A bistar $B_{n,m}$ is a Lipschitz graph.
Proof. Let G be a bistar $B_{n,m}$.
Then $|V(G)| = n+m+2$ and $|E(G)| = n+m+1$.
Let f be a labeling of G with the following conditions.

Vertex labeling

- Each vertex in the left pendant vertex is assigned a unique label $1,2,\ldots,n$, sequentially from left to right.
- Each vertex in the right pendant vertex is assigned a unique label $n+1, n+2, \ldots, n+m$, sequentially from left to right.
- Assign $n+m+1$ to the vertex adjacent with all the left pendant vertices.
- Assign $n+m+2$ to the vertex adjacent with all the right pendant vertices.

Edge lebeling

$$f(v_i v_j) = n+m+2 + \min\{f(v_i), f(v_j)\}$$

for all egde $v_i v_j \in E(G)$.
Let $x = v_i v_j$ and $y = v_k v_l$ be any two egdes in the graph.
Then

$$\begin{aligned} |f(x) - f(y)| &= \left|\left(n + m + 2 + \min\{f(v_i), f(v_j)\}\right) - (n + m + 2 + \min\{f(v_k), f(v_l)\})\right| \\ &= \left|\min\{f(v_i), f(v_j)\} - \min\{f(v_k), f(v_l)\}\right| \\ &= \left|x_f - y_f\right| \end{aligned}$$

From the above two discussion, we have

$$|f(x) - f(y)| \leq L \cdot \left|x_f - y_f\right|$$

for all $x, y \in E(G)$ with $L = 1$.
Hence $B_{n,m}$ is a Lipschitz graph.

4 Concluding Remarks

This study has successfully investigated the application of Lipschitz labeling to several structured graph families. We have developed a constructive algorithm to generate valid Lipschitz labelings for the twinkling star graph with rays of length $2n(n \geq 1)$, the triangular graph Λ_n, and the bistar graph $B_{n,m}$. The feasibility of this labeling scheme was demonstrated through a detailed illustrative example, confirming the theoretical results.

Our work solidifies the position of Lipschitz labeling as a viable and interesting graph labeling method, particularly for tree-like structures. By establishing these labelings for more complex star-based graphs, we have extended the known boundaries of this problem. Several intriguing questions remain open, such as characterizing all tree families that admit a Lipschitz labeling or determining the labelability of graph products. We hope this research serves as a catalyst for further exploration into the properties and applications of Lipschitz labeling across a wider spectrum of graph theory.

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