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PRODUCT OF TOTAL AND RANGE FUZZY EDGE LABELING OF CERTAIN GRAPH FAMILIES

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Abstract

Fuzzy graph labeling deals with the assignment of membership values to the vertices and edges of a graph under prescribed constraints. In this paper, a product of total and range fuzzy edge labeling is introduced. A graph $G = (V, E)$ is said to admit this labeling if there exist bijections $\rho(v): V \to [0,1]$ and $\delta(uv): E \to [0,1]$ such that $\delta(uv) < \min\{\rho(u), \rho(v)\}$ and

$$\delta(uv) = |\rho(u)^2 - \rho(v)^2|$$

for all $uv \in E$. Graphs satisfying these conditions are identified as product of total and range fuzzy edge labeling graphs. Using this framework, it is shown that the path graph, cycle graph, friendship graph, triangular snake graph, triangular book, generalized butterfly graph, ladder graph and coconut tree graph admit such a labeling.

Key words: Fuzzy edge labeling, product of total and range fuzzy edge labelling graph.

1 Introduction

Graph theory was introduced by the Swiss mathematician Leonhard Euler in 1736 to model the Königsberg bridge problem and to show that the problem has no solution. Later, graph labeling emerged as an important area of graph theory with the introduction of vertex labeling by [8] A. Rosa in 1966. [17] In recent years, fuzzy set theory, proposed by Zadeh, has played a vital role in extending classical graph concepts to situations involving uncertainty and imprecision. [3]-9] The combination of graph theory and fuzzy set theory led to the development of fuzzy graphs, where vertices and edges are assigned membership values from the interval [0,1]. Fuzzy graph labeling has since become an active area of research due to its wide applicability in real-world problems involving vagueness and partial information. Motivated by these developments, various fuzzy labeling schemes have been proposed and studied. In this paper, a new labeling scheme called product of total and range fuzzy edge labeling is introduced, where the vertex and edge membership functions satisfy specific algebraic and ordering constraints. Several classes of graphs are examined under this definition, and it is shown that path graphs, cycle graphs, friendship graphs, triangular book graphs, and generalized butterfly graphs admit product of total and range fuzzy edge labeling. Throughout this paper, only simple undirected graphs are considered.

2 Definitions of Terminologies Used in the Study

Definition 2.1. [12] A graph $G = (\rho, \delta)$ is said to be a fuzzy labeling graph if $\rho: V \to [0,1]$ and $\delta: V \times V \to [0,1]$ is bijective such that the membership value of edges and vertices are distinct and $\delta(u, v) < \rho(u) \wedge \rho(v)$ for all $u, v \in V$.

Definition 2.2. [10] The friendship graph Fr_n is a collection of n triangles with a common vertex. It is a planar undirected graph with $2n + 1$ vertices and $3n$ edges constructed by joining n copies of the cycle graph C_3 with a common vertex.

Definition 2.3. [18] A Triangular snake T_n is obtained from a path $v_1, v_2, v_3, \dots, v_n$ by joining v_i and v_{i+1} to a new vertex u_i for $1 \leq i \leq n - 1$.

Definition 2.4. [10] The Generalized butterfly graph, denoted by BF_n, is obtained by inserting adjoining vertices to every wing with the assumption that the sum of inserting vertices to every wing is the same.

Definition 2.5. [19] The ladder graph L_n is defined by $L_n = P_n \times K_2$, where P_n is a path with n vertices, $\times$ denotes the Cartesian product, and K_2 is a complete graph with two vertices.

Definition 2.6. A graph $G = (V, E)$ admits a product of total and range fuzzy edge labeling if there exist bijections $\rho(v): V \to [0,1]$ and $\delta(uv): E \to [0,1]$ satisfying

$$\delta(uv) < \min\{\rho(u), \rho(v)\}$$

and

$$\delta(uv) = |\rho(u)^2 - \rho(v)^2|$$

for all $uv \in E$.

3 Computation of Fuzzy Labelling under Uncertainty using the Product of Total-Range Method

Theorem 3.1. Every fuzzy path P_n admits a product of total and range fuzzy edge labelling for $n \geq 2$.

Proof. The vertex set of P_n is

$$V(P_n) = \{u_1, u_2, \dots, u_n\}$$

and the edge set is

$$E(P_n) = \{u_i u_{i+1} \mid 1 \leq i \leq n - 1\}.$$

Clearly, $|V(P_n)| = n$ and $|E(P_n)| = n - 1$.

The membership function for the vertices,

$$\rho(v): V \to [0,1]$$

is defined as

$$\rho(u_i) = \frac{n + i}{2n}, 1 \leq i \leq n.$$

The membership function for the edges,

$$\delta(uv): E(P_n) \to [0,1],$$

is defined by

$$\delta(e_i) = |\rho(u_i)^2 - \rho(u_{i+1})^2|, 1 \leq i \leq n - 1.$$

Hence, P_n produces a total and range fuzzy graph. The membership value for each edge satisfies

$$\delta(uv) < \min\{\rho(u), \rho(v)\}.$$

Example: Consider a fuzzy path graph P_5.

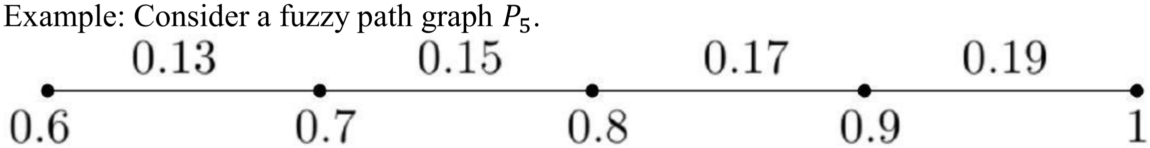

Figure 1:

Theorem 3.2. The fuzzy cycle graph C_n admits a product of total and range fuzzy edge labelling graph for $n \geq 3$.

Proof. Let C_n has cycle n vertices and n edges. Define the vertex set

$$V(C_n) = \{u_1, u_2, \dots, u_n\}$$

and the edge set

$$E(C_n) = \{u_i u_{i+1} \mid 1 \le i \le n-1\} \cup \{u_n u_1\}.$$

Clearly,

$$|V(C_n)| = n \text{ and } |E(C_n)| = n.$$

The membership function for the vertices,

$$\rho(v): V(C_n) \to [0,1]$$

Case (i): n is even.

$$\rho(u_i) = \begin{cases} 1, & \text{if } i = 1, \\ \dfrac{2n-i}{2n}, & \text{if } i = 2,4,\dots,n-2, \\ \dfrac{2n-i+2}{2n}, & \text{if } i = 3,5,\dots,n-1, \\ \dfrac{n+3}{2n+1}, & \text{if } i = n. \end{cases}$$

Case (ii): n is odd.

$$\rho(u_i) = \begin{cases} 1, & \text{if } i = 1, \\ \dfrac{2n-i}{2n}, & \text{if } i = 2,4,\dots,n-1, \\ \dfrac{2n-i+2}{2n}, & \text{if } i = 3,5,\dots,n-2, \\ \dfrac{n+3}{2n+1}, & \text{if } i = n. \end{cases}$$

The membership function for the Edges,

$$\delta(uv): E(C_n) \to [0,1]$$

$$\delta(e_i) = |\rho(u_i)^2 - \rho(u_{i+1})^2|.$$

Thus, the product of total and range fuzzy edge labelling graph satisfies

$$\delta(uv) < \min\{\rho(u), \rho(v)\}.$$

Example

Consider the fuzzy cycle C_6.

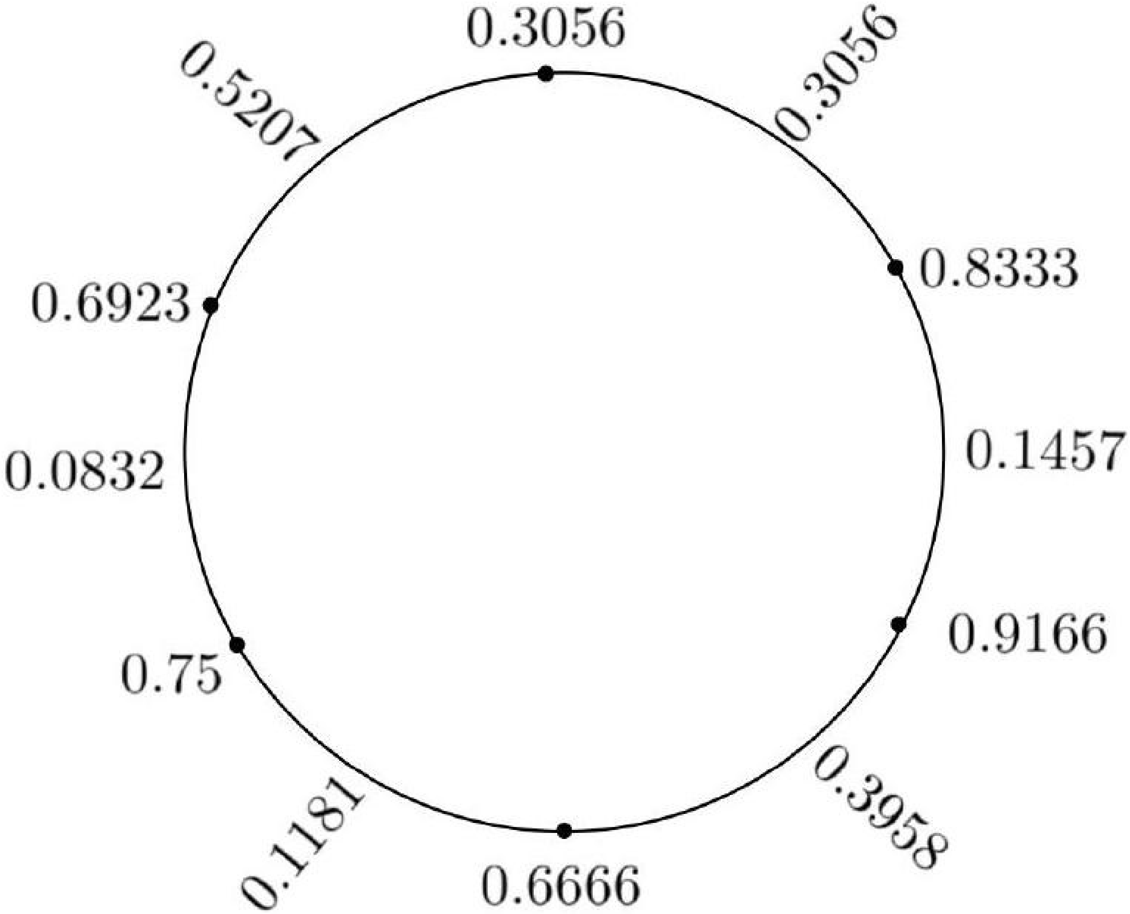

Figure 2:

The vertex labels are defined as

$$\rho(u_1) = 1, \quad \rho(u_2) = 0.8333, \quad \rho(u_3) = 0.9166$$
$$\rho(u_4) = 0.6666, \quad \rho(u_5) = 0.75, \quad \rho(u_6) = 0.6923$$

The edge labels are defined as

$$\delta(e_1) = 0.3056, \quad \delta(e_2) = 0.1457, \quad \delta(e_3) = 0.3958$$
$$\delta(e_4) = 0.1181, \quad \delta(e_5) = 0.0832, \quad \delta(e_6) = 0.5207.$$

Theorem 3.3. The fuzzy friendship graph F_n admits a product of total and range fuzzy edge labelling graph.

Proof. Let F_n be a fuzzy friendship graph consisting of n triangles sharing a common vertex. Define the vertex set

$$V(F_n) = \{w_0, u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$$

where w_0 is the central vertex and the edge set is

$$E(F_n) = \{w_0u_i \mid 1 \leq i \leq n\} \cup \{w_0v_i \mid 1 \leq i \leq n\} \cup \{u_iv_i \mid 1 \leq i \leq n\}.$$

With

$$|V| = 2n + 1 \text{ and } |E| = 3n$$

The membership function for the vertices is

$$\rho(u): V(F_n) \to [0,1].$$
$$\rho(w_0) = \frac{n}{n+5}, 1 \le i \le 2n+1.$$
$$\rho(u_i) = \frac{2n-(2i-1)}{2n+1}, 1 \le i \le n.$$
$$\rho(v_i) = \frac{2n-2i}{2n+1}, 1 \le i \le n.$$

The membership function for the edges is

$$\delta(uv): E(F_n) \to [0,1]$$
$$\delta(e_i) = |\rho(w_0)^2 - \rho(u_i)^2|$$
$$\delta(f_i) = |\rho(w_0)^2 - \rho(v_i)^2|$$
$$\delta(g_i) = |\rho(u_i)^2 - \rho(v_i)^2|$$

Where

$$\delta(e_i) = |w_0 u_i|, \delta(f_i) = |w_0 v_i|, \delta(g_i) = |u_i v_i|$$

Hence, the product of total and range fuzzy edge graph membership value satisfies

$$\delta(uv) < \min\{\rho(u), \rho(v)\}.$$

Example: Consider a fuzzy friendship graph F_n with $n = 7$.

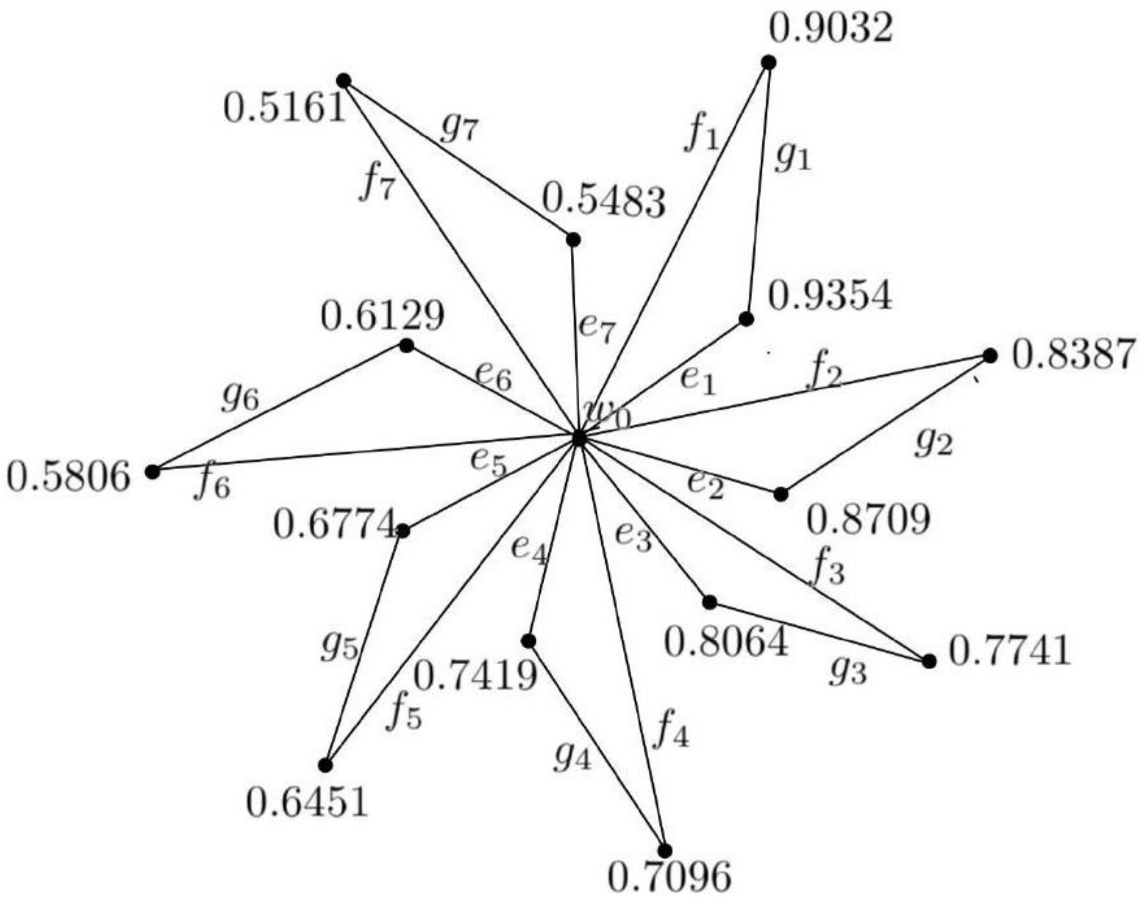

Figure 3:

The vertex labels are

$$\rho(w_0) = 0.06,$$
$$\rho(u_1) = 0.9354, \rho(v_1) = 0.9032,$$
$$\rho(u_2) = 0.8709, \rho(v_2) = 0.8387,$$
$$\rho(u_3) = 0.8064, \rho(v_3) = 0.7741,$$
$$\rho(u_4) = 0.7419, \rho(v_4) = 0.7096,$$
$$\rho(u_5) = 0.6774, \rho(v_5) = 0.6451,$$
$$\rho(u_6) = 0.6129, \rho(v_6) = 0.5806,$$
$$\rho(u_7) = 0.5483, \rho(v_7) = 0.5161.$$

The edge labels are

$$\delta(e_1) = 0.5149, \quad \delta(f_1) = 0.4557, \quad \delta(g_1) = 0.0575,$$
$$\delta(e_2) = 0.3984, \quad \delta(f_2) = 0.3434, \quad \delta(g_2) = 0.0550,$$
$$\delta(e_3) = 0.2902, \quad \delta(f_3) = 0.2392, \quad \delta(g_3) = 0.0510,$$
$$\delta(e_4) = 0.1904, \quad \delta(f_4) = 0.1435, \quad \delta(g_4) = 0.0469,$$
$$\delta(e_5) = 0.0988, \quad \delta(f_5) = 0.0561, \quad \delta(g_5) = 0.0427,$$
$$\delta(e_6) = 0.0156, \quad \delta(f_6) = 0.0229, \quad \delta(g_6) = 0.0385,$$
$$\delta(e_7) = 0.0593, \quad \delta(f_7) = 0.0936, \quad \delta(g_7) = 0.0342.$$

Theorem 3.4. The fuzzy triangular snake graph T_n admits a product of total and range fuzzy edge labelling for all $n \geq 2$.

Proof. Let T_n be the fuzzy triangular snake graph. The vertex set and edge set of T_n are defined as follows:

$$V(T_n) = \{v_i: 1 \leq i \leq n\} \cup \{u_i: 1 \leq i \leq n-1\},$$
$$E(T_n) = \{v_i v_{i+1}, v_i u_i, u_i v_{i+1}: 1 \leq i \leq n-1\}.$$

Clearly,

$$|V(T_n)| = 2n - 1 \text{ and } |E(T_n)| = 3n - 3.$$

Define the vertex membership function

$$\rho(u): V(T_n) \to (0,1)$$

by

$$\rho(v_i) = \frac{2i-1}{4n}, 1 \leq i \leq n, \rho(u_i) = \frac{2i}{4n}, 1 \leq i \leq n-1.$$

Clearly,

$$0 < \rho(x) < 1 \text{ for all } x \in V(T_n),$$

and all vertex membership values are distinct.
Define the edge membership function

$$\delta(uv): E(T_n) \to (0,1).$$

For $1 \leq i \leq n-1$,

$$\delta(p_i) = |\rho(v_i)^2 - \rho(u_i)^2|$$
$$\delta(q_i) = |\rho(u_i)^2 - \rho(v_{i+1})^2|$$
$$\delta(r_i) = |\rho(v_i)^2 - \rho(v_{i+1})^2|$$

where $e_i = v_i u_i, e_i' = u_i v_{i+1}$ and $f_i = v_i v_{i+1}$.
Since $\rho(u) \neq \rho(v)$ for all $uv \in E(T_n)$, we have

$$\delta(uv) > 0.$$

Also,

$$\delta(uv) < \min\{\rho(u), \rho(v)\}, \forall uv \in E(T_n).$$

Hence, the fuzzy triangular snake graph T_n admits a product of total and range fuzzy edge labelling .
□

Example: Consider the fuzzy triangular snake graph T_n.

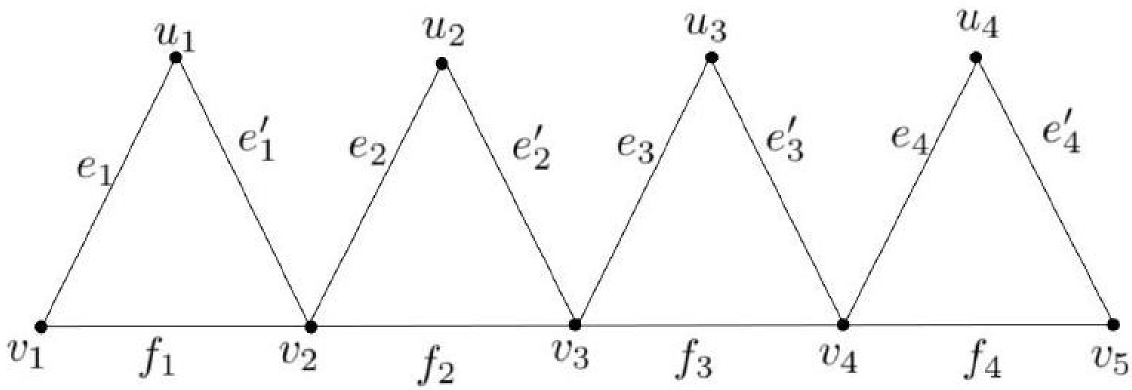

Figure 4:

The vertex labels are defined as,

$$\rho(v_1) = 0.05, \quad \rho(u_1) = 0.10,$$
$$\rho(v_2) = 0.15, \quad \rho(u_2) = 0.20,$$
$$\rho(v_3) = 0.25, \quad \rho(u_3) = 0.30,$$
$$\rho(v_4) = 0.35, \quad \rho(u_4) = 0.40,$$
$$\rho(v_5) = 0.45.$$

The edge labels are defined as,

$$\delta(f_1) = 0.02, \quad \delta(e_1) = 0.0075, \quad \delta(e_1') = 0.0125,$$
$$\delta(f_2) = 0.04, \quad \delta(e_2) = 0.0175, \quad \delta(e_2') = 0.0225,$$
$$\delta(f_3) = 0.06, \quad \delta(e_3) = 0.0275, \quad \delta(e_3') = 0.0325,$$
$$\delta(f_4) = 0.08, \quad \delta(e_4) = 0.0375, \quad \delta(e_4') = 0.0425,$$

Theorem 3.5. The fuzzy triangular book graph TB_n admits a Product of Total and Range Fuzzy Edge Labeling.

Proof. Let TB_n be a fuzzy triangular book graph with vertex set

$$V(TB_n) = \{u, v, w_1, w_2, \dots, w_n\}$$

and edge set

$$E(TB_n) = \{uv, uw_i, vw_i : 1 \le i \le n\}.$$

clearly,

$$|V(TB_n)| = n + 2, |E(TB_n)| = 1 + 2n.$$

Define the vertex membership function

$$\rho(v): V(TB_n) \to [0,1]$$

by

$$\rho(u) = \frac{n}{3n}$$
$$\rho(v) = \frac{n+1}{3n}$$
$$\rho(w_i) = \frac{n-i+2}{2n+1}, 1 \le i \le n.$$

Define the edge membership function

$$\delta: E(TB_n) \to [0,1]$$

by

$$\delta(e_i) = |\rho(u_i)^2 - \rho(v_i)^2|$$
$$\delta(f_i) = |\rho(u_i)^2 - \rho(w_i)^2|$$
$$\delta(g_i) = |\rho(v_i)^2 - \rho(w_i)^2|$$

Where

$$\delta(e_i) = |uv|, \delta(f_i) = |uw_i|, \delta(g_i) = |vw_i|$$

is satisfied. Hence, the fuzzy triangular book graph B_n admits a Product of Total and Range Fuzzy Edge Labeling.

Example:

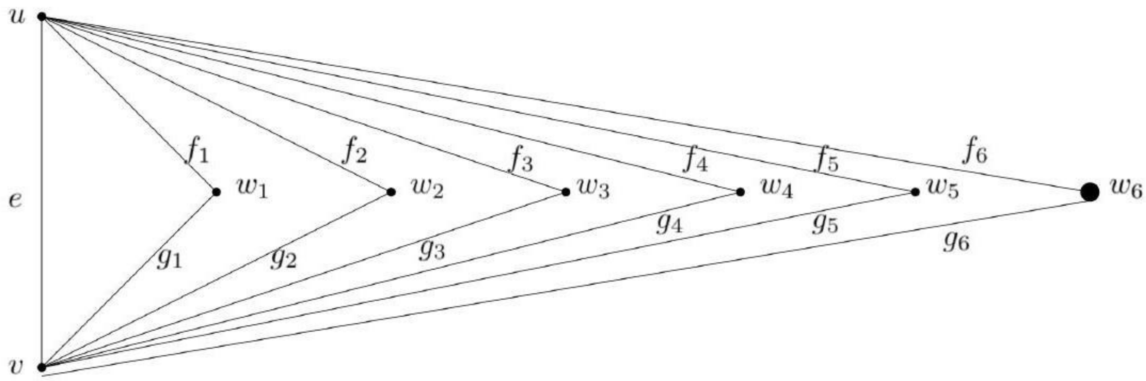

Figure 5:

The vertex labels are defined as,

$$\rho(u) = 0.3333, \quad \rho(w_1) = 0.5384,$$
$$\rho(v) = 0.3888, \quad \rho(w_2) = 0.4615,$$
$$\rho(w_3) = 0.3846,$$
$$\rho(w_4) = 0.3076,$$
$$\rho(w_5) = 0.2307,$$
$$\rho(w_6) = 0.1538.$$

The edge labels are defined as,

$$\delta(e) = 0.0400$$
$$\delta(f_1) = 0.1787, \quad \delta(g_1) = 0.1387$$
$$\delta(f_2) = 0.1018, \quad \delta(g_2) = 0.0618$$
$$\delta(f_3) = 0.0368, \quad \delta(g_3) = 0.0032$$
$$\delta(f_4) = 0.0164, \quad \delta(g_4) = 0.0565$$
$$\delta(f_5) = 0.0578, \quad \delta(g_5) = 0.0979$$
$$\delta(f_6) = 0.0874, \quad \delta(g_6) = 0.1275$$

Theorem 3.6. A generalized butterfly fuzzy graph BF_n admits a product of total and range fuzzy edge labeling for $n \geq 2$.

Proof. Let BF_n be generalized butterfly fuzzy graph BF_n. Define the vertex set,

$$V(BF_n) = \{v_i \mid 0 \leq i \leq 2n\}$$

and the edge set of BF_n be,

$$E(BF_n) = \{v_i v_{i+1} \mid i = 1,2,\dots,n-1,n+1,\dots,2n-1\} \cup \{v_0 v_i \mid i = 1,2,\dots,2n\}.$$

Where

$$|V(G)| = 2n + 1 \text{ and } |E(G)| = 4n - 2.$$

The membership function for the vertices,

$$\rho: V(BF_n) \to [0,1]$$
$$\rho(v_0) = \frac{1}{2n+3},$$
$$\rho(v_i) = \frac{i+1}{(2n+1)^2}, 1 \leq i \leq 2n.$$

The membership function for the edges,

$$\delta: E(BF_n) \to [0,1]$$
$$\delta(e_i) = |\rho(v_0)^2 - \rho(v_i)^2|$$
$$\delta(f_i) = |\rho(v_i)^2 - \rho(v_{i+1})^2|$$

Thus, the product of total and range fuzzy edge labeling graph satisfies

$$\delta(uv) < \min\{\rho(u), \rho(v)\}, \forall uv \in E(T_n).$$

Hence proved.

Example:

Consider a generalized butterfly fuzzy graph with $n = 5$.

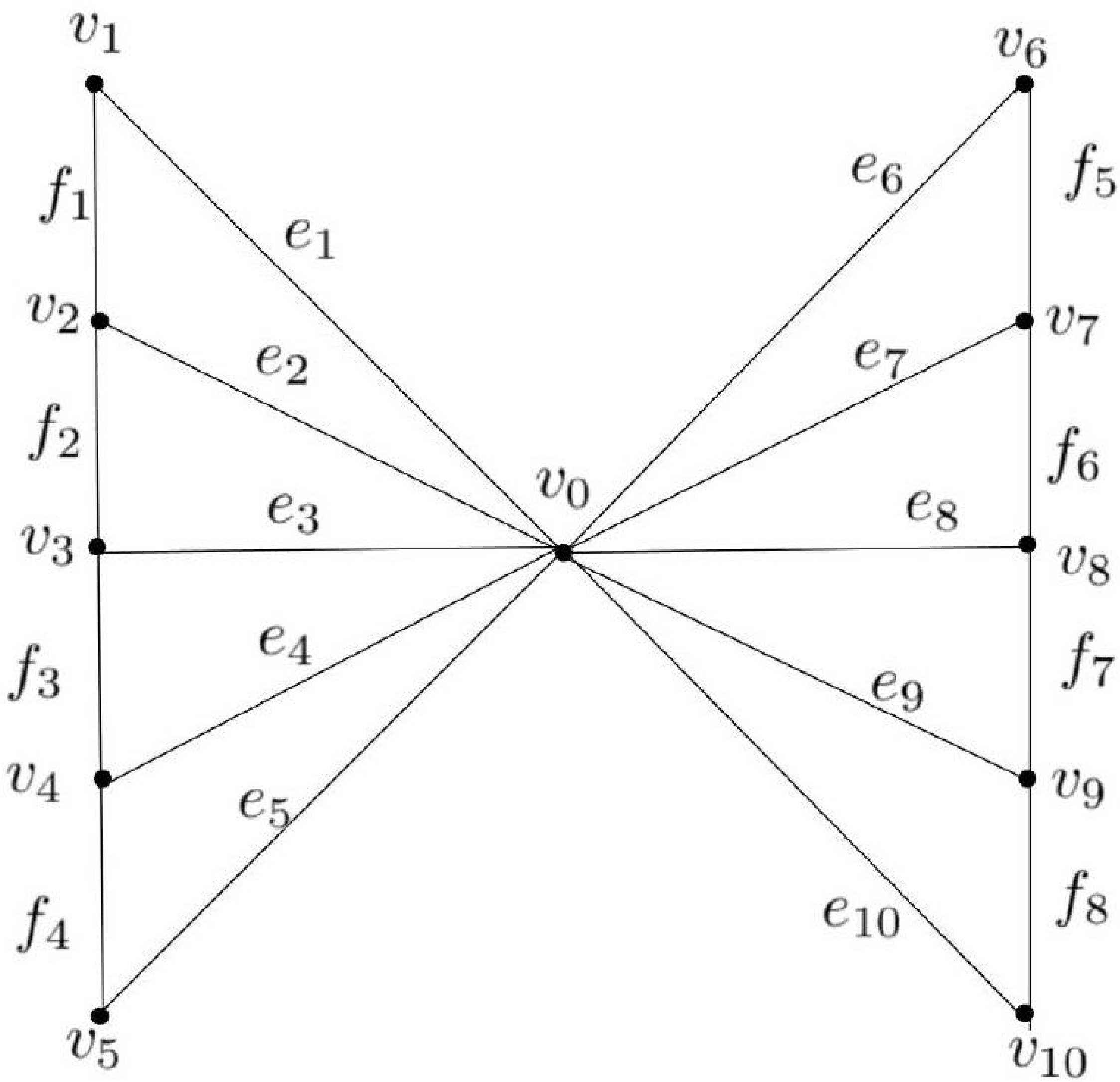

Figure 6:

The vertex labels are defined as,

$$\rho(v_1) = 0.0118, \quad \rho(v_6) = 0.0414,$$
$$\rho(v_2) = 0.0177, \quad \rho(v_7) = 0.0473,$$
$$\rho(v_3) = 0.0236, \quad \rho(v_8) = 0.0532,$$
$$\rho(v_4) = 0.0295, \quad \rho(v_9) = 0.05917,$$
$$\rho(v_5) = 0.0355, \quad \rho(v_{10}) = 0.0650.$$

The edge labels are defined as,

$\delta(e_1) = 0.0057,\quad \delta(e_6) = 0.0041$
$\delta(e_2) = 0.0056,\quad \delta(e_7) = 0.0036$
$\delta(e_3) = 0.0053,\quad \delta(e_8) = 0.0030$
$\delta(e_4) = 0.0050,\quad \delta(e_9) = 0.0024$
$\delta(e_5) = 0.0046,\quad \delta(e_{10}) = 0.0016$
$\delta(f_1) = 0.00017,\quad \delta(f_5) = 0.00052$
$\delta(f_2) = 0.00024,\quad \delta(f_6) = 0.00059$
$\delta(f_3) = 0.00031,\quad \delta(f_7) = 0.00066$
$\delta(f_4) = 0.00039,\quad \delta(f_8) = 0.00073$

Theorem 3.7. Let $L_n = P_n \odot P_2$ be the fuzzy ladder graph on $2n$ vertices. Then L_n admits a product of total and range fuzzy edge labeling.

Proof. Let

$$V(L_n) = \{u_i, v_i \mid 1 \le i \le n\},$$

where u_i and v_i represent the vertices on the two parallel paths of the ladder. Define the vertex membership function

$$\rho: V(L_n) \to (0,1)$$

by

$$\rho(u_i) = \frac{2i-1}{4n}, \rho(v_i) = \frac{2i}{4n}, 1 \le i \le n.$$

Clearly, ρ is bijective and $0 < \rho(v) < 1$ for all $v \in V(L_n)$. The edge set of L_n is

$$E(L_n) = \{u_i u_{i+1}, v_i v_{i+1} \mid 1 \le i \le n-1\} \cup \{u_i v_i \mid 1 \le i \le n\}.$$

Define the edge membership function

$$\delta: E(L_n) \to (0,1)$$

by

$$\delta(e_i) = |\rho(u_i)^2 - \rho(u_{i+1})^2|$$
$$\delta(e_i') = |\rho(v_i)^2 - \rho(v_{i+1})^2|$$
$$\delta(g_i) = |\rho(u_i)^2 - \rho(v_i)^2|$$

For every edge $uv \in E(L_n)$,

$$\delta(uv) = |\rho(u)^2 - \rho(v)^2| < \min\{\rho(u), \rho(v)\}.$$

Hence, the fuzzy ladder graph L_n admits a product of total and range fuzzy edge labeling.

Example

Consider the fuzzy ladder graph L_n for $n = 9$.

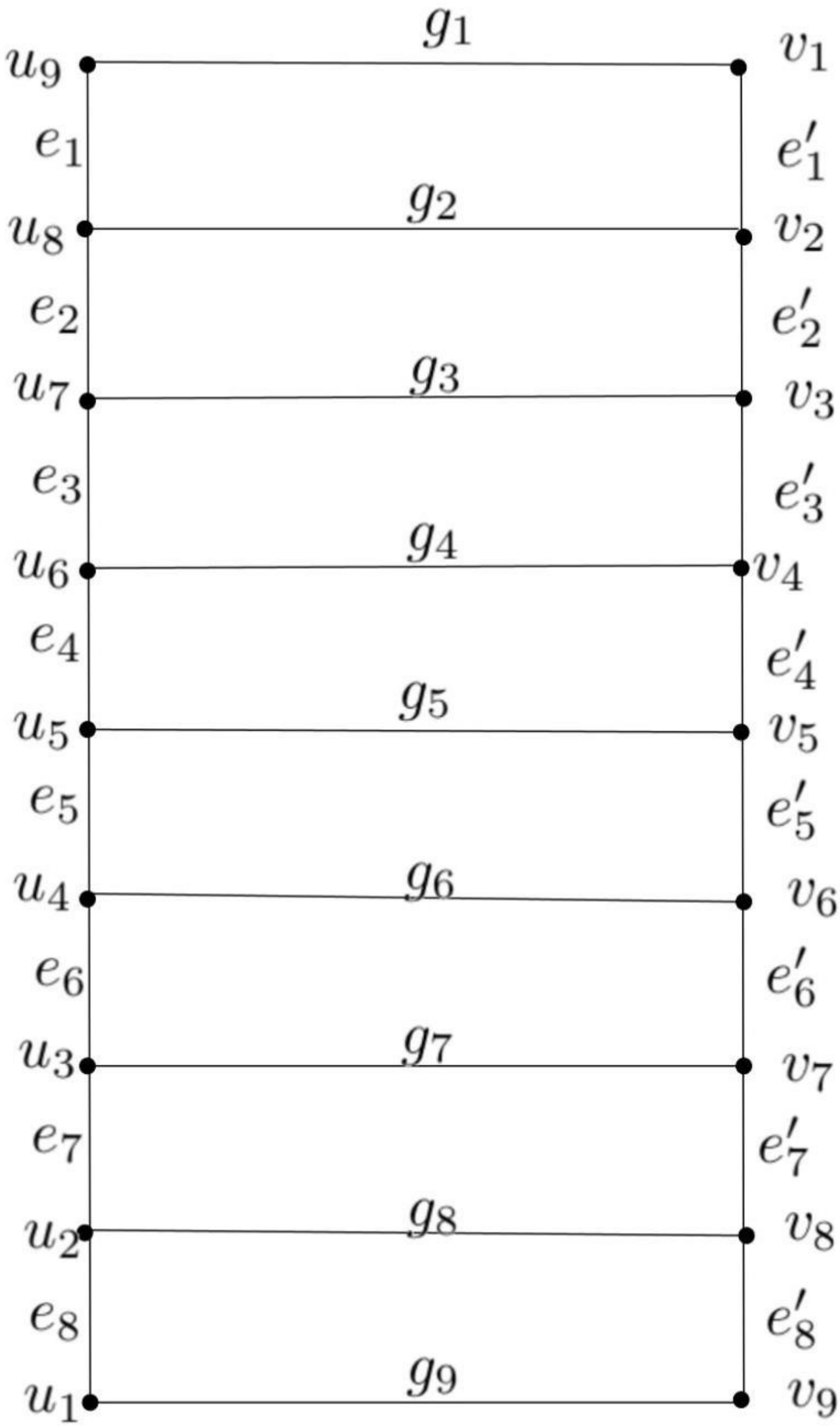

Figure 7:

The vertex labels are defined as,

$$\rho(u_1) = 0.0277, \quad \rho(v_1) = 0.0555,$$
$$\rho(u_2) = 0.0833, \quad \rho(v_2) = 0.1111,$$
$$\rho(u_3) = 0.1388, \quad \rho(v_3) = 0.1666,$$
$$\rho(u_4) = 0.1944, \quad \rho(v_4) = 0.2222,$$
$$\rho(u_5) = 0.25, \quad \rho(v_5) = 0.2777,$$
$$\rho(u_6) = 0.3055, \quad \rho(v_6) = 0.3333,$$
$$\rho(u_7) = 0.3611, \quad \rho(v_7) = 0.3888,$$
$$\rho(u_8) = 0.4166, \quad \rho(v_8) = 0.4444,$$
$$\rho(u_9) = 0.4722, \quad \rho(v_9) = 0.5.$$

The edge labels are defined as,

$$\delta(e_1) = 0.0061, \delta(e'_1) = 0.0092,$$
$$\delta(e_2) = 0.0123, \delta(e'_2) = 0.0154,$$
$$\delta(e_3) = 0.0185, \delta(e'_3) = 0.0216,$$
$$\delta(e_4) = 0.0247, \delta(e'_4) = 0.0277,$$
$$\delta(e_5) = 0.0308, \delta(e'_5) = 0.0339,$$
$$\delta(e_6) = 0.0370, \delta(e'_6) = 0.0400,$$
$$\delta(e_7) = 0.0431, \delta(e'_7) = 0.0463,$$
$$\delta(e_8) = 0.0494, \delta(e'_8) = 0.0525.$$
$$\delta(g_1) = 0.0023,$$
$$\delta(g_2) = 0.0054,$$
$$\delta(g_3) = 0.0084,$$
$$\delta(g_4) = 0.0115,$$
$$\delta(g_5) = 0.0146,$$
$$\delta(g_6) = 0.0177,$$
$$\delta(g_7) = 0.0307,$$
$$\delta(g_8) = 0.0239,$$
$$\delta(g_9) = 0.0270.$$

Theorem 3.8. The fuzzy coconut tree graph CT_n admits a product of total and range fuzzy edge labeling for all $n \geq 2$.

Proof. Let CT_n be the fuzzy coconut tree graph. The vertex set and edge set of CT_n are defined as follows:

$$V(CT_n) = \{u_i: 1 \leq i \leq n\} \cup \{w_i: 1 \leq i \leq n\},$$
$$E(CT_n) = \{u_i w_i: 1 \leq i \leq n\} \cup \{u_i u_{i+1}: 1 \leq i \leq n-1\}.$$

Clearly,

$$|V(CT_n)| = 2n \text{ and } |E(CT_n)| = 2n - 1$$

Define the vertex membership function

$$\rho: V(CT_n) \to (0,1)$$

by

$$\rho(u_i) = \frac{i}{4n}, 1 \leq i \leq n$$
$$\rho(w_i) = \frac{n+i}{4n}, 1 \leq i \leq n$$

Clearly,

$$0 < \rho(x) < 1 \text{ for all } x \in V(CT_n),$$

The membership function for the edges,

$$\delta(uv): E(F_n) \to [0,1].$$
$$\delta(e_i) = |\rho(w_i)^2 - \rho(u_i)^2|.$$
$$\delta(f_i) = |\rho(w_i)^2 - \rho(w_i + 1)^2|.$$

Where

$$\delta(e_i) = |w_i u_i|, \delta(f_i) = |w_i w_i + 1|,$$

Hence fuzzy coconut tree graph CT_n admits a product of total and range fuzzy edge labeling. □

Example: Consider a fuzzy graph $G = CT_{n,m}$ with $n = 7$ and $m = 4$

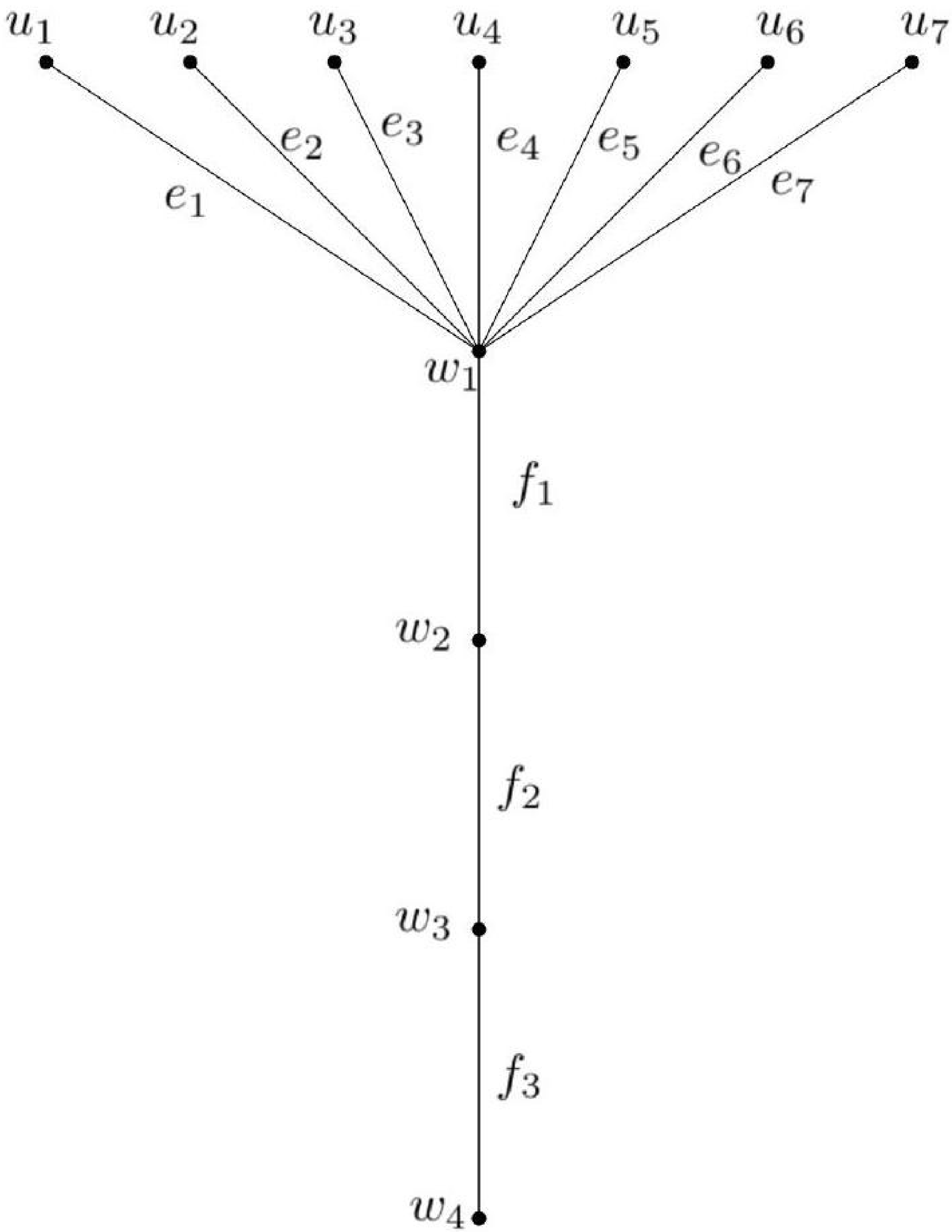

Figure 8:

The vertex labels are defined as,

$$\rho(u_1) = 0.0178, \rho(w_1) = 0.0681,$$
$$\rho(u_2) = 0.0535, \rho(w_2) = 0.1136,$$
$$\rho(u_3) = 0.0892, \rho(w_3) = 0.1590,$$
$$\rho(u_4) = 0.125, \rho(w_4) = 0.2045,$$
$$\rho(u_5) = 0.1607,$$
$$\rho(u_6) = 0.1964,$$
$$\rho(u_7) = 0.2321.$$

The edge labels are defined as,

$$\delta(e_1) = 0.0043, \delta(f_1) = 0.0082,$$
$$\delta(e_2) = 0.0017, \delta(f_2) = 0.0123,$$
$$\delta(e_3) = 0.0033, \delta(f_3) = 0.0165,$$
$$\delta(e_4) = 0.0109,$$
$$\delta(e_5) = 0.0211,$$
$$\delta(e_6) = 0.039,$$
$$\delta(e_7) = 0.0492.$$

4 Conclusion

The Product of Total and Range Fuzzy Edge Labeling provides a more comprehensive way to represent relationships in fuzzy graphs by simultaneously capturing the interaction strength and variation between vertex memberships. This dual characterization enhances the expressive power of fuzzy graph models compared to existing labeling schemes. The selection of various standard graph families such as path, cycle, friendship graph, triangular snake graph,triangular book graph, generalized butterfly graph, ladder graph, ladder graph and coconut tree graph ensures the structural diversity and mathematical robustness of the proposed labeling. These graph families serve as fundamental models in graph theory and real-world networks, thereby highlighting the theoretical significance and potential applicability of the proposed labeling scheme in areas involving uncertainty.

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