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# THE Y-INDEX FOR DOUBLE CORONA OF GRAPHS RELATED TO THE DIFFERENT SUBDIVISION GRAPHS

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**Abstract:** Graph operations play a significant role in generating complex molecular and network structures from the simpler graphs. The computation of degree-based topological indices of such derived structure is an essential problem in chemical graph theory due to their applications in QSAR/QAPR studies. Among them, the Y index is a recently studied degree-based invariant that extends the family of Zagreb-type indices and captures structural information of graphs more efficiently. In this paper, we generalize the concepts of double corona graphs associated with subdivision-related graph operations and derive exact analytical expressions for the Y-index of these constructions. Closed-form formulae are obtained for the subdivision $\mathrm{S}, \mathrm{R}, \mathrm{Q}, \mathrm{T}$ and total graph based double corona products in terms of standard graph invariants and degree parameters. The obtained result extends the theoretical framework of degree-based indices under complex graph operations such as PANAM dendritic polymers and provide useful mathematical tools for modeling intricate molecular networks.

**Keywords:** Zagreb indices; F-index; S-index; corona product graphs; graph operations.

AMS Subject Classification: 05C09, 05C76

## 1. Introduction

Graph theory is one of the fundamental branch of Mathematics. It provides, a strong framework for analyzing complex structures and obtain efficient formulation and solution of problems arising in various scientific disciplines.

Chemical graph theory is a specialized branch of graph theory, that facilitates the theoretical analysis of molecular structures, and their properties. Where molecular structures are modeled by graphs with vertices representing atoms and edges representing chemical bonds. A topological indices play a central role as numerical descriptors that explain essential structural information of molecular graphs and are widely used in QSAR/QSPR studies. These topological indices analyse the structure of any finite and are based on mathematical equations.

Different kinds of topological indices exist (i.e)., degree-based topological indices[8], distance-based topological indices and counting-related topological indices[9].

In this paper we consider a simple connected graph H. The vertex set and edge set of H are defined by $\mathrm{V}(\mathrm{H})$ and $\mathrm{E}(\mathrm{H})$ respectively. The degree of a vertex ' a ' in H is defined as number of edges incident with ' a ' and it is denoted by $\mathrm{d}(\mathrm{a})$ or $d_H(a)$. The distance between the vertices a and b in $\mathrm{V}(\mathrm{H})$ is denoted as $\mathrm{d}(\mathrm{a}, \mathrm{b})$ be the number of edges in a shortest path connecting the vertices 'a' and 'b'. For understanding more notations and terminologies we can refer [17].

In 1947, Wiener[20] introduced Wiener index, $W(H) = \frac{1}{2}\sum_{(a,b)} d(a,b)$.
Where $( a, b )$ is the order pair of vertices in H and $\mathrm{d(a,b)}$ is the distance of the vertices $\mathrm{a-b}$ in H .

The Zagreb indices play a main role in degree-based topological indices. It was introduced by Gutman and Trinajestic[7] to study the total $\pi$-electron energy $(\mathcal{E})$ of carbon atoms in 1972 , and it was denoted as

$$\begin{aligned} M_1(H) &= \sum_{v\in V(H)} d_H(v)^2 \\ &= \sum_{uv\in E(H)} [d_H(u) + d_H(v)] \text{ and} \\ M_2(H) &= \sum_{uv\in E(H)} d_H(u)d_H(v) \end{aligned}$$

The geometric - arithmetics index (GA)[21][19] of graph $H$ is defined as,

$$GA(H) = \sum_{uv\in E(H)} \frac{2\sqrt{d_u d_v}}{d_u + d_v}$$

The atomic bond connectivity index [ABC] of a graph $H$ is denoted as[10]

$$ABC(H) = \sum_{uv\in E(H)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}$$

In 2005 Li and Zheng [12, 3, 2, 13] introduced the first general Zagreb index as

$$M_1^{\alpha+1}(H) = \sum_{v\in V(H)} d_H^{\alpha+1}(v) = \sum_{uv\in E(H)} d_H^{\alpha}(u) + d_H^{\alpha}(v)$$

In 2009, MH Khalifeha, H. Yousefi Azaria and AR Ashraf[11]is defined the first and second hyper Zagreb index if a connected graph H by [16], [14]

$$HM_1(H) = \sum_{uv\in E(H)} (d_H(u) + d_H(v))^2$$
$$HM_2(H) = \sum_{uv\in E(H)} (d_H(u)d_H(v))^2$$

In 2013, Ranjini et al. first introduced the redefined Zagreb index and it is denoted by [6]

$$\mathrm{Re}ZM(G) = \sum_{uv\in E(H)} d_H(u)d_H(v)[d_H(u) + d_H(v)]$$

Furtula and Gutman[5] in 2015 introduced Forgotten topological index (F-index) which is defined as [4]

$$F(H) = \sum_{v\in V(H)} d_H(v)^3$$
$$= \sum_{uv\in E(H)} [d_H(u)^2 + d_H(v)^2]$$

In 2018, S. Ghobadi and M. Ghorbaninejad[18] defined a new distance based of Zagreb indices named Forgotten topological index or hyper-F-index defined as

$$HF(H) = \sum_{uv\in E(H)} [d_H^2(u) + d_H^2(v)]^2$$

In 2020 A Alameri, et al.[1] introduced Y index as

$$Y(H) = \sum_{v\in V(H)} d_H(v)^4$$
$$= \sum_{uv\in E(H)} (d_H(u)^3 + d_H(v)^3)$$

In 2021, S Nagarajan, G Kayalvizhi and G Priyadharsini[15] introduced S index as

$$M_5(H) = S(H) = \sum_{v\in V(H)} d_H(v)^5$$
$$= \sum_{uv\in E(H)} [d_H(u)^4 + d_H(v)^4]$$

## 2. Preliminaries

A connected graph $H = (V(H), E(H))$ has no self loop and no parallel edges, where vertex set be $V(H)$ and edge set be $E(H)$. The number of vertices in $V(H)$ is $n$ and i.e., $|V(H)| = n$. The edge set contains $m$ edges i.e., $|E(H)| = m$.

Let us consider $H_1$ and $H_2$ be two simple connected graph and their vertex sets and edge sets be $V(H_1), V(H_2)$ and $E(H_1), E(H_2)$ respectively.

$$\text{Let } \begin{matrix} |V(H_1)| = n_1 & \qquad |V(H_2)| = n_2 \\ |E(H_1)| = m_1 & \qquad |E(H_2)| = m_2 \end{matrix}$$

The subdivision graph of a graph $H$ is obtained by inserting an extra vertex into each edge of $H$ and is denoted by $S(H)$

The graph $R(H)$ is obtained from $H$ by inserting an additional vertex into each edge of $H$ and joining each additional vertex to the end vertices of the corresponding edge of $H$. $Q(H)$ is a graph derived from $H$ by adding a new vertex to each edge of $H$, then joining with edges those pairs of new vertices on adjacent edges of $H$.

The Total graph $T(H)$ is derived from $H$ by inserting an new vertex to each edge of $H$, then joining each new vertex to the end vertices of the corresponding edge and joining with edges those pairs of new vertices on adjacent edges of $H$.

In this paper, we are studying $Y$ index for double corona of graphs related to the different subdivision graphs.Let $Y = \{S, R, Q, T\}$.

Definition 2.1. Let $H_1$ and $H_2$ be two simple graphs with $n_1, n_2$ number of vertices and $m_1, m_2$ number of edges. The corona of $H_1$ and $H_2$ is obtain by taking one copy of $H_1$ and $n_1$ copies of $H_2$ and then connecting each $i^{\text{th}}$ vertex of $H_1$ to every vertex in the $i^{\text{th}}$ copy of $H_2$ where $i = 1,2, \dots n_1$ and is denoted by $H_1 \circ H_2$

Definition 2.2. Suppose that $H$ be a graph of $n$ number of vertices and $m$ number of edges. The $Y$-double corona graph of $H, H_1$ and $H_2$ is denoted by $\{H^Y \otimes (H_1^{\cdot}, H_2^{\circ})\}$ and derived by taking one copy $Y(H), n$ copies of $H_1$ and $m$ copies of $H_2$ then joining $i^{\text{th}}$ old-vertex of $Y(H)$ to every vertex in the $i^{\text{th}}$ copy of $H_1$ and the $j^{\text{th}}$ new-vertex of $Y(H)$ to every vertex in the $j^{\text{th}}$ copy of $\mathrm{H}_2$.

Now if we replace $Y$ by $S, R, Q, T$ then we get $S$-double corona, $R$-double corona, $Q$-double corona, and $T$-double corona respectively.

## 3. Main Results

### 3.1. $S$-double corona graph compute by $Y$-index.

Let S(H)-double corona graph is shown as

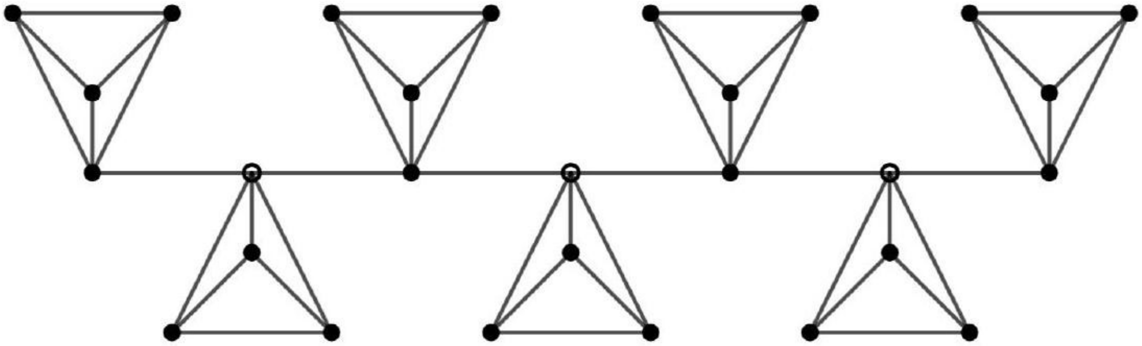

Figure 1. $P_4^S \otimes (C_3^{\cdot}, C_3^{\circ})$

Lemma 3.1. All the vertex degree of ( $H^S \otimes (H_1^{\cdot}, H_2^{\circ})$ ) are given by

$$d_{\left(H^S \otimes (H_1^{\cdot}, H_2^{\circ})\right)}(v) = \begin{cases} d_H(v_i) + n_1, \text{ if } v_i \in V(H), i = 1,2, \dots n \\ 2 + n_2, \text{ if } u_i v_j = w \in I(H), i, j = 1,2, \dots n, i \neq j \\ d_{H_1}\left(v_i^j\right) + 1, \text{ if } v_i^j \in V(H_1), i = 1,2, \dots n_1 \text{ and } j = 1,2 \dots, n \\ d_{H_2}\left(v_i^j\right) + 1, \text{ if } v_i^j \in V(H_2), i = 1,2, \dots n_2 \text{ and } j = 1,2 \dots, m \end{cases}$$

Where $I(H)$ denotes the set of all new vertices which are inserted in $H$.

Theorem 3.2. For three connected graphs, $H, H_1$ and $H_2$ we get

$$\begin{aligned} Y\left(H^S \otimes (H_1^{\cdot}, H_2^{\circ})\right) = & Y(H) + nY(H_1) + mY(H_2) + 4n_1F(H) + 4nF(H_1) \\ & +4mF(H_2) + 6n_1^2M_1(H) + 6nM_1(H_1) + 6mM_1(H_2) \\ & +8(n_1^3m + nm_1 + mm_2) + nn_1(1 + n_1^3) \\ & +m(n_2 + (2 + n_2)^4) \end{aligned}$$

Proof. By definition of $Y(H)$ we have,

| $v_i \in V(H)$ | $u_iv_j = w \in I(H)$ | $v_i^j \in V(H_1)$ | $v_i^j \in V(H_2)$ |
|---|---|---|---|
| $i = 1,2, \dots n$ | $i, j = 1,2, \dots n$<br>$i \neq j$ | $i = 1,2, \dots n_1$ and<br>$j = 1,2 \dots, n$ | $i = 1,2, \dots n_2$ and<br>$j = 1,2 \dots, m$ |
| $d_H(v_i) + n_1$ | $2 + n_2$ | $d_{H_1}\left(v_i^j\right) + 1$ | $d_{H_2}\left(v_i^j\right) + 1$ |

$$\begin{aligned} Y\left(H^S \otimes (H_1^{\cdot}, H_2^{\circ})\right) &= \sum_{v \in \left(H^S \otimes (H_1^{\cdot}, H_2^{\circ})\right)} d_{\left(H^S \otimes (H_1^{\cdot}, H_2^{\circ})\right)}(v)^4 \\ &= \sum_{v \in V(H)} d_{\left(H^S \otimes (H_1^{\cdot}, H_2^{\circ})\right)}(v)^4 + \sum_{v \in I(H)} d_{\left(H^S \otimes (H_1^{\cdot}, H_2^{\circ})\right)}(v)^4 \\ &+n \sum_{v \in V(H_1)} d_{\left(H^S \otimes (H_1^{\cdot}, H_2^{\circ})\right)}(v)^4 + m \sum_{v \in V(H_2)} d_{\left(H^S \otimes (H_1^{\cdot}, H_2^{\circ})\right)}(v)^4 \\ &= \sum_{v \in V(H)} (d_H(v) + n_1)^4 + \sum_{v \in I(H)} (2 + n_2)^4 \\ &+n \sum_{v \in V(H_1)} \left(d_{H_1}(v) + 1\right)^4 + m \sum_{v \in V(H_2)} \left(d_{H_2}(v) + 1\right)^4 \end{aligned}$$

$$\begin{aligned}
&= \sum_{v\in V(H)} d_H(v)^4 + 4n_1 \sum_{v\in V(H)} d_H(v)^3 \\
&+6n_1^2 \sum_{v\in V(H)} d_H(v)^2 + 4n_1^3 \sum_{v\in V(H)} d_H(v) \\
&+n_1^4 \sum_{v\in V(H)} 1 + m(2+n_2)^4 + n \sum_{v\in V(H_1)} d_{H_1}(v)^4 \\
&+4n \sum_{v\in V(H_1)} d_{H_1}(v)^3 + 6n \sum_{v\in V(H_1)} d_{H_1}(v)^2 \\
&+4n \sum_{v\in V(H_1)} d_{H_1}(v) + n \sum_{v\in V(H_1)} 1 \\
&+m \sum_{v\in V(H_2)} d_{H_2}(v)^4 + 4m \sum_{v\in V(H_2)} d_{H_2}(v)^3 \\
&+6m \sum_{v\in V(H_2)} d_{H_2}(v)^2 + 4m \sum_{v\in V(H_2)} d_{H_2}(v) + m \sum_{v\in V(H_2)} 1
\end{aligned}$$

$$\begin{aligned}
Y\big(H^S \otimes (H_1^{\cdot}, H_2^{\circ})\big) &= Y(H) + 4n_1F(H) + 6n_1^2M_1(H) + 4n_1^3(2m) \\
&+nn_1^4 + m(2+n_2)^4 + nY(H_1) + 4nF(H_1) \\
&+6nM_1(H_1) + 4n(2m_1) + nn_1 + mY(H_2) \\
&+4mF(H_2) + 6mM_1(H_2) + 4m(2m_2) + mn_2
\end{aligned}$$

$$\begin{aligned}
Y\big(H^S \otimes (H_1^{\cdot}, H_2^{\circ})\big) &= Y(H) + nY(H_1) + mY(H_2) + 4n_1F(H) \\
&+4nF(H_1) + 4mF(H_2) + 6n_1^2M_1(H) \\
&+6nM_1(H_1) + 6mM_1(H_2) \\
&+8(n_1^3m + nm_1 + mm_2) + nn_1(1+n_1^3) \\
&+m(n_2 + (2+n_2)^4)
\end{aligned}$$

**Example 3.3.**

$$\begin{aligned}
Y[P_l^s \otimes (C_m^{\cdot}, C_n^{\circ})] &= (l-1)[8m^3 + 57n + (2+n)^4] \\
&+lm(m^3 + 24m + 113) \\
&-4m(9m+14) + 24n(n-1) \\
&+16l - 30 && ,l \geq 2, n, m \geq 3 \\
Y[C_l^s \otimes (P_m^{\cdot}, C_n^{\circ})] &= lm(m^3 + 8m^2 + 24m + 113) \\
&+l[(2+n)^4 + 81n + 114] && ,l, n \geq 3, m \geq 2 \\
Y[P_l^s \otimes (P_m^{\cdot}, P_n^{\circ})] &= (l-1)(8n-130) + (l-1)[8m^3 + (2+n)^4] \\
&+l(80m - 114) + \mathrm{lm}(1+m^3) - 30 && ,l, n, m \geq 2
\end{aligned}$$

$$\begin{aligned}
Y[C_l^s \otimes (C_m^{\cdot}, C_n^{\circ})] &= l(2+n)^4 + 16l + 81ln \\
&+113lm + 24lm^2 + 8lm^3 + lm^4 && ,l, m \geq 3
\end{aligned}$$

### 3.2. $R$-double corona graph obtained using $Y$-index.

Let $R(H)$-double corona is shown as

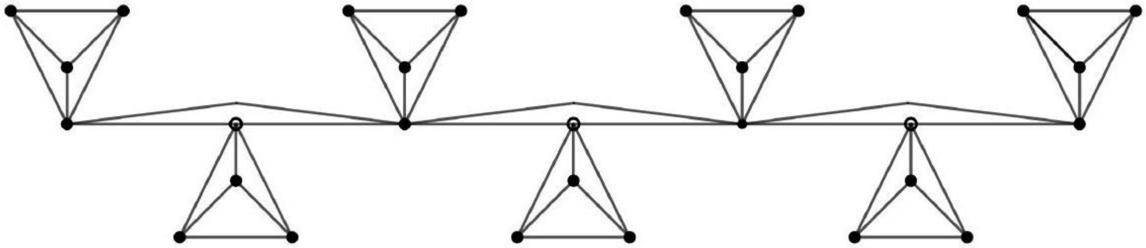

Figure 2. $P_4^R \otimes (C_3^{\cdot}, C_3^{\circ})$

Lemma 3.4. All the vertex degrees of ( $H^R \otimes (H_1^{\cdot}, H_2^{\circ})$ ) are given by

$$d_{\left(H^R \otimes (H_1, H_2^{\circ})\right)}(v) = \begin{cases} 2d_H(v_i) + n_1, \text{ if } v_i \in V(H), i = 1,2, \dots n \\ 2 + n_2, \text{ if } u_i v_j = w \in I(H), i,j = 1,2, \dots n, i \neq j \\ d_{H_1}\left(v_i^j\right) + 1, \text{ if } v_i^j \in V(H_1), i = 1,2, \dots n_1 \text{ and } j = 1,2 \dots, n \\ d_{H_2}\left(v_i^j\right) + 1, \text{ if } v_i^j \in V(H_2), i = 1,2, \dots n_2 \text{ and } j = 1,2 \dots, m \end{cases}$$

Theorem 3.5. If $H, H_1$ and $H_2$ are three connected graphs, then we get

$$\begin{aligned} Y\left(H^R \otimes (H_1^{\cdot}, H_2^{\circ})\right) = & 16Y(H) + nY(H_1) + mY(H_2) + 32n_1F(H) + 4nF(H_1) \\ & + 4mF(H_2) + 24n_1^2M_1(H) + 6nM_1(H_1) + 6mM_1(H_2) \\ & + 16n_1^3m + nn_1^4 + m(2 + n_2)^4 + 8nm_1 + nn_1 + 8mm_2 \\ & + mn_2 \end{aligned}$$

Proof. By definition of $R(H)$ we have,

| $v_i \in V(H)$ | $u_i v_j = w \in I(H)$ | $v_i^j \in V(H_1)$ | $v_i^j \in V(H_2)$ |
|---|---|---|---|
| $i = 1,2, \dots n$ | $i,j = 1,2, \dots n, i \neq j$ | $i = 1,2, \dots, n_1$ and $j = 1,2, \dots n$ | $i = 1,2, \dots, n_2$ and $j = 1,2, \dots m$ |
| $2d_H(v_i) + n_1$ | $2 + n_2$ | $d_{H_1}\left(v_i^j\right) + 1$ | $d_{H_2}\left(v_i^j\right) + 1$ |

$$
\begin{aligned}
Y\big(H^R \otimes (H_1^{\cdot}, H_2^{\circ})\big) &= \sum_{v \in \left(H^R \otimes (H_1^{\cdot}, H_2^{\circ})\right)} d_{\left(H^R \otimes (H_1^{\cdot}, H_2^{\circ})\right)}(v)^4 \\
&= \sum_{v \in V(H)} d_{\left(H^R \otimes (H_1^{\cdot}, H_2^{\circ})\right)}(v)^4 + \sum_{v \in I(H)} d_{\left(H^S \otimes (H_1^{\cdot}, H_2^{\circ})\right)}(v)^4 \\
&\quad + n \sum_{v \in V(H_1)} d_{\left(H^R \otimes (H_1^{\cdot}, H_2^{\circ})\right)}(v)^4 + m \sum_{v \in V(H_2)} d_{\left(H^R \otimes (H_1^{\cdot}, H_2^{\circ})\right)}(v)^4 \\
&= \sum_{v \in V(H)} (2d_H(v) + n_1)^4 + \sum_{v \in I(H)} (2 + n_2)^4 \\
&\quad + n \sum_{v \in V(H_1)} \left(d_{H_1}(v) + 1\right)^4 + m \sum_{v \in V(H_2)} \left(d_{H_2}(v) + 1\right)^4 \\
Y\big(H^R \otimes (H_1^{\cdot}, H_2^{\circ})\big) &= 16 \sum_{v \in V(H)} d_H(v)^4 + 32 n_1 \sum_{v \in V(H)} d_H(v)^3 \\
&\quad + 24 n_1^2 \sum_{v \in V(H)} d_H(v)^2 + 8 n_1^3 \sum_{v \in V(H)} d_H(v) + n_1^4 \sum_{v \in V(H)} 1 \\
&\quad + m(2 + n_2)^4 + n \sum_{v \in V(H_1)} d_{H_1}(v)^4 + 4n \sum_{v \in V(H_1)} d_{H_1}(v)^3 \\
&\quad + 6n \sum_{v \in V(H_1)} d_{H_1}(v)^2 + 4n \sum_{v \in V(H_1)} d_{H_1}(v) + n \sum_{v \in V(H_1)} 1 \\
&\quad + m \sum_{v \in V(H_2)} d_{H_2}(v)^4 + 4m \sum_{v \in V(H_2)} d_{H_2}(v)^3 \\
&\quad + 6m \sum_{v \in V(H_2)} d_{H_2}(v)^2 + 4m \sum_{v \in V(H_2)} d_{H_2}(v) + m \sum_{v \in V(H_2)} 1. \\
Y\big(H^R \otimes (H_1^{\cdot}, H_2^{\circ})\big) &= 16Y(H) + nY(H_1) + mY(H_2) + 32 n_1 F(H) \\
&\quad + 4nF(H_1) + 4mF(H_2) + 24 n_1^2 M_1(H) \\
&\quad + 6nM_1(H_1) + 6mM_1(H_2) + 16 n_1^3 m + n n_1^4 \\
&\quad + m(2 + n_2)^4 + 8nm_1 + nn_1 + 8mm_2 + mn_2
\end{aligned}
$$

□

**Example 3.6.**

$$
\begin{aligned}
Y[P_l^R \otimes (C_m^{\cdot}, C_n^{\circ})] &= (l-1)[81n + (2+n)^4 + 16m^3) \\
&\quad + lm(337 + 96m + m^2) \\
&\quad - 16m(28 - 9m) - 480 && , l \geq 2, n, m \geq 3 \\
Y[C_l^R \otimes (P_m^{\cdot}, C_n^{\circ})] &= lm(377 + 96m + 16m^2) \\
&\quad + l(66 + (2+n)^4) && , l, n \geq 3, m \geq 2 \\
&\quad + 81ln + nm^4
\end{aligned}
$$

$$Y[P_l^R \otimes (P_m^{\cdot}, P_n^{\circ})] = lm^4 + 16lm^3 + 96lm^2$$
$$+337lm - 16m^3 - 144m^2$$
$$-448m - 126l + (l-1)(81n - 130)$$
$$+(l-1)(2+n)^4 - 480 \qquad , l, n, m \geq 2$$
$$Y[C_l^R \otimes (C_m^{\cdot}, C_n^{\circ})] = lm^4 + 16lm^3 + 96lm^2$$
$$+337lm + l(2+n)^4 + 256l + 81ln \qquad , l, m, n \geq 3$$

### 3.3. $Y$-index compute $Q$-double corona graphs.

Let $Q(H)$-double corona of $H, H_1$ and $H_2$ is shown in the figure. We define the degree of all the vertices of $Q$-double corona graphs in the following lemma.

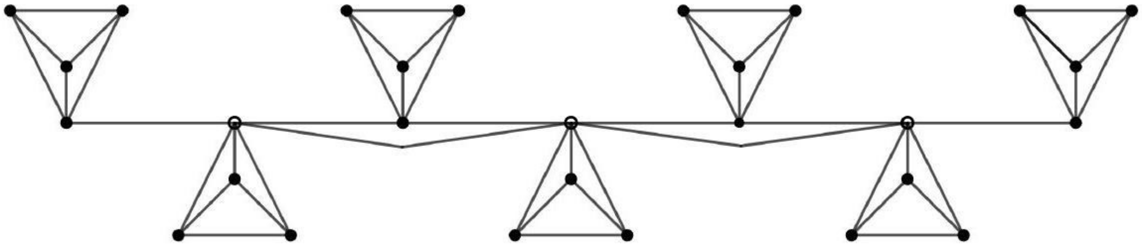

Figure 3. $H^Q \otimes (H_1^{\cdot}, H_2^{\circ})$

Lemma 3.7. The degree of all the vertices of ( $H^Q \otimes (H_1^{\cdot}, H_2^{\circ})$ ) are given by

$$d_{\left(H^Q \otimes (H_1^{\cdot}, H_2^{\circ})\right)}(v) = \begin{cases} d_H(v_i) + n_1, \text{ if } v_i \in V(H), i = 1,2, \dots n \\ d_H(u_i) + d_H(v_j) + n_2, \text{ if } u_i v_j = w \in I(H), i,j = 1,2, \dots n, i \neq j \\ d_{H_1}(v_i^j) + 1, \text{ if } v_i^j \in V(H_1), i = 1,2, \dots n_1 \text{ and } j = 1,2 \dots, n \\ d_{H_2}(v_i^j) + 1, \text{ if } v_i^j \in V(H_2), i = 1,2, \dots n_2 \text{ and } j = 1,2 \dots, m \end{cases}$$

Theorem 3.8. For three connected graphs $H, H_1$ and $H_2$ we get

$$Y\left(H^Q \otimes (H_1^{\cdot}, H_2^{\circ})\right) = Y(H) + 4n_2Y(H) + nY(H_1) + mY(H_2) + 4n_1F(H) + 4nF(H_1)$$
$$+4mF(H_2) + 6n_1^2M_1(H_1) + 4n_2^3M_1(H) + 6nM_1(H_1) + 6mM_1(H_2)$$
$$+M_5(H) + 4Z_{3,1}(H) + 6HM_2(H) + 6n_2^2HM_1(H) + 12n_2Z_eZM(H)$$
$$+8(mn_1^3 + nm_1 + mm_2) + nn_1(n_1^3 + 1) + mn_2(n_2^3 + 1)$$

Proof. By definition of $Q(H)$ we have,

| $v_i \in V(H)$ | $u_i v_j = w$ $\in I(H)$ | $v_i^j \in V(H_1)$ | $v_i^j \in V(H_2)$ |
|---|---|---|---|
| $i = 1,2, \dots n$ | $i,j = 1,2, \dots n, i$ $\neq j$ | $i =$ $1,2, \dots n_1$ and $j = 1,2 \dots, n$ | $i = 1,2, \dots n_2$ and $j = 1,2 \dots, m$ |
| $d_H(v_i) + n_1$ | $d_H(u_i)$ $+ d_H(v_j) + n_2$ | $d_{H_1}(v_i^j) + 1$ | $d_{H_2}(v_i^j) +$ |

$$Y\left(H^{Q}\otimes(H_1^{\cdot},H_2^{\circ})\right)=\sum_{v\in\left(H^{Q}\otimes(H_1^{\cdot},H_2^{\circ})\right)} d_{\left(H^{Q}\otimes(H_1^{\cdot},H_2^{\circ})\right)}(v)^4$$

$$=\sum_{v\in V(H)} d_{\left(H^{Q}\otimes(H_1^{\cdot},H_2^{\circ})\right)}(v)^4+\sum_{v\in I(H)} d_{\left(H^{Q}\otimes(H_1^{\cdot},H_2^{\circ})\right)}(v)^4$$

$$+n\sum_{v\in V(H_1)} d_{\left(H^{Q}\otimes(H_1^{\cdot},H_2^{\circ})\right)}(v)^4+m\sum_{v\in V(H_2)} d_{\left(H^{Q}\otimes(H_1^{\cdot},H_2^{\circ})\right)}(v)^4$$

$$=\sum_{v\in V(H)} (d_H(v)+n_1)^4+\sum_{uv\in E(H)} (d_H(u)+d_H(v)+n_2)^4$$

$$+n\sum_{v\in V(H_1)} \left(d_{H_1}(v)+1\right)^4+m\sum_{v\in V(H_2)} \left(d_{H_2}(v)+1\right)^4$$

$$=\sum_{v\in V(H)} [d_H(v)^4+4d_H(v)^3n_1+6d_H(v)^2n_1^2+4d_H(v)n_1^3+n_1^4]$$

$$+\sum_{uv\in E(H)} \{[(d_H(u))+d_H(v)]^4+4[d_H(u)+d_H(v)]^3n_2$$

$$+6[d_H(u)+d_H(v)]^2n_2^2+4[d_H(u)+d_H(v)]n_2^3+n_2^4\}$$

$$+n\sum_{v\in V(H_1)} \left[d_{H_1}(v)^4+4d_{H_1}(v)^3+6d_{H_1}(v)^2+4d_{H_1}(v)+1\right]$$

$$+m\sum_{v\in V(H_2)} \left[d_{H_2}(v)^4+4d_{H_2}(v)^3+6d_{H_2}(v)^2+4d_{H_2}(v)+1\right].$$

$$=\sum_{v\in V(H)} d_H(v)^4+4n_1\sum_{v\in V(H)} d_H(v)^3+6n_1^2\sum_{v\in V(H)} d_H(v)^2$$

$$+4n_1^3\sum_{v\in V(H)} d_H(v)+n_1^4\sum_{v\in V(H)} 1+\sum_{uv\in E(H)} [d_H(u)^4+d_H(v)^4]$$

$$+4\sum_{uv\in E(H)} [d_H(u)^3d_H(v)+d_H(u)d_H(v)^3]+6\sum_{uv\in E(H)} d_H(u)^2d_H(v)^2$$

$$+4n_2\sum_{uv\in E(H)} [d_H(u)^3+d_H(v)^3]+12n_2\sum_{uv\in E(H)} d_H(u)d_H(v)[d_H(u)$$

$$+d_H(v)]+6n_2^2\sum_{uv\in E(H)} (d_H(u)+d_H(v))^2$$

$$+4n_2^3\sum_{uv\in E(H)} [d_H(u)+d_H(v)]+n_2^4\sum_{uv\in E(H)} 1$$

$$+n\sum_{v\in V(H_1)} d_{H_1}(v)^4+4n\sum_{v\in V(H_1)} d_{H_1}(v)^3$$

$$+6n\sum_{v\in V(H_1)} d_{H_1}(v)^2+4n\sum_{v\in V(H_1)} d_{H_1}(v)+n\sum_{v\in V(H_1)} 1$$

$$+m\sum_{v\in V(H_2)} d_{H_2}(v)^4+4m\sum_{v\in V(H_2)} d_{H_2}(v)^3+6m\sum_{v\in V(H_2)} d_{H_2}(v)^2$$

$$+4m\sum_{v\in V(H_2)} d_{H_2}(v)+m\sum_{v\in V(H_2)} 1$$

$$\begin{aligned} Y\left(H^{Q} \otimes (H_1^{\cdot}, H_2^{\circ})\right) = {} & Y(H) + 4n_2Y(H) + nY(H_1) + mY(H_2) \\ & +4n_1F(H) + 4nF(H_1) + 4mF(H_2) + 6n_1^2M_1(H) \\ & +4n_2^3M_1(H) + 6nM_1(H_1) + 6mM_1(H_2) + M_5(H) \\ & +4Z_{3,1}(H) + 6HM_2(H) + 6n_2^2HM_1(H) + 12n_2Z_eZM(H) \\ & +8(mn_1^3 + nm_1 + mm_2) + nn_1(n_1^3 + 1) + mn_2(n_2^3 + 1) \end{aligned}$$

**Example 3.9.**

$$\begin{aligned} Y\left[P_l^{Q} \otimes (C_m^{\cdot}, C_n^{\circ})\right] = {} & (4l - 6)(6m^2 + 4n^3) + (l - 1) \\ & (8m^3 + n^4) + \mathrm{lm}(m^3 + 1) \\ & +(16l - 30)(6n^2 + 16n) \\ & +112lm + 81ln - 55n \\ & +56m + 144l - 636 \qquad , l \geq 2, n, m \geq 3 \end{aligned}$$

$$\begin{aligned} Y\left[C_l^{Q} \otimes (P_m^{\cdot}, C_n^{\circ})\right] = {} & l(m^4 + n^4) + 8l(m^3 + 2n^3) \\ & +24l(m^2 + 4n^2) + 337ln \\ & +113lm + 142 \qquad , l, m, n \geq 3 \end{aligned}$$

$$\begin{aligned} Y\left[P_l^{Q} \otimes (P_m^{\cdot}, P_n^{\circ})\right] = {} & l(m^4 + n^4) + 8l(m^3 + 2n^3) \\ & +24l(m^2 + 4n^2) + 113lm \\ & +337ln - 56m - 36m^2 \\ & -8m^3 + 12l - 633n \\ & -180n^2 - 24n^3 + n^4 - 506 \qquad , l, n, m \geq 2 \end{aligned}$$

$$\begin{aligned} Y\left[C_l^{Q} \otimes (C_m^{\cdot}, C_n^{\circ})\right] = {} & l(m^4 + n^4) + 8l(m^3 + 2n^3) \\ & +24l(m^2 + 4n^2) + 113lm \\ & +337ln + 272l \qquad , l, m, n \geq 3 \end{aligned}$$

### 3.4. $Y$ index for $T$-double corona graph.

Let $H, H_1$ and $H_2$ are three simple connected graphs

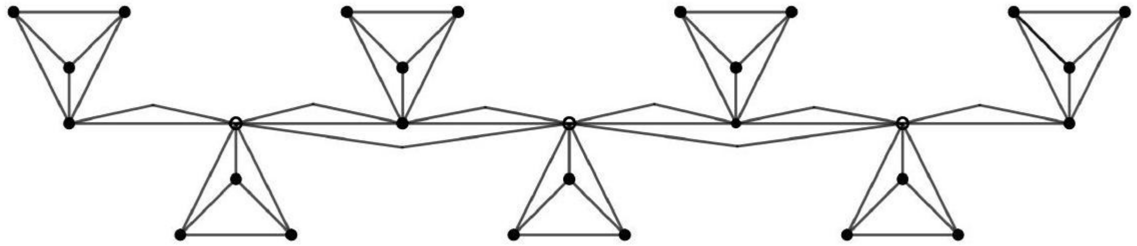

Figure 4. $H^{T} \otimes (H_1^{\cdot}, H_2^{\circ}))$

Lemma 3.10. The degree of all the vertices of ( $H^{T} \otimes (H_1^{\cdot}, H_2^{\circ})$ ) are given by

$$d_{\left(H^T\otimes(H_1^{\cdot},H_2^{\circ})\right)}(v)=\begin{cases}2d_H(v_i)+n_1, \text{ if } v_i\in V(H), i=1,2,\ldots n\\ d_H(u_i)+d_H(v_j)+n_2, \text{ if } u_iv_j=w\in I(H), i,j=1,2,\ldots n, i\neq j\\ d_{H_1}(v_i^j)+1, \text{ if } v_i^j\in V(H_1), i=1,2,\ldots n_1 \text{ and } j=1,2\ldots,n\\ d_{H_2}(v_i^j)+1, \text{ if } v_i^j\in V(H_2), i=1,2,\ldots n_2 \text{ and } j=1,2\ldots,m\end{cases}$$

Theorem 3.11. If $H, H_1$ and $H_2$ are three connected graphs then

$$\begin{aligned}Y\left(H^T\otimes(H_1^{\cdot},H_2^{\circ})\right)=&16Y(H)+32n_1F(H)+24n_1^2M_1(H)+16mn_1^3\\ &+nn_1^4+M_5(H)+4Z_{3,1}(H)+6HM_2(H)\\ &+4n_2Y(H)+12n_2ZeZM(H)+6n_2^2HM_1(H)\\ &+4n_2^3M_1(H)+mn_2^4+nY(H_1)+4nF(H_1)\\ &+6nM_1(H_1)+8m_1n+nn_1+mY(H_2)\\ &+4mF(H_2)+6mM_1(H_2)+8mm_2+mn_2\end{aligned}$$

Proof. By the definition of $T(H)$ we have,

| $v_i\in V(H)$ | $u_iv_j=w\in I(H)$ | $v_i^j\in V(H_1)$ | $v_i^j\in V(H_2)$ |
|---|---|---|---|
| $i=1,2,\ldots n$ | $i,j=1,2,\ldots n, i\neq j$ | $i=1,2,\ldots n_1$ and $j=1,2\ldots,n$ | $i=1,2,\ldots n_2$ and $j=1,2\ldots,m$ |
| $2d_H(v_i)+n_1$ | $d_H(u_i)+d_H(v_j)+n_2$ | $d_{H_1}(v_i^j)+1$ | $d_{H_2}(v_i^j)+1$ |

$$\begin{aligned}Y\left(H^T\otimes(H_1^{\cdot},H_2^{\circ})\right)&=\sum_{v\in\left(H^T\otimes(H_1^{\cdot},H_2^{\circ})\right)}d_{\left(H^T\otimes(H_1^{\cdot},H_2^{\circ})\right)}(v)^4\\ &=\sum_{v\in V(H)}d_{\left(H^T\otimes(H_1^{\cdot},H_2^{\circ})\right)}(v)^4+\sum_{v\in I(H)}d_{\left(H^T\otimes(H_1^{\cdot},H_2^{\circ})\right)}(v)^4\\ &+n\sum_{v\in V(H_1)}d_{\left(H^T\otimes(H_1^{\cdot},H_2^{\circ})\right)}(v)^4+m\sum_{v\in V(H_2)}d_{\left(H^T\otimes(H_1^{\cdot},H_2^{\circ})\right)}(v)^4\\ &=16\sum_{v\in V(H)}d_H(v)^4+nn_1^4+32n_1\sum_{v\in V(H)}d_H(v)^3\\ &+24n_1^2\sum_{v\in V(H)}d_H(v)^2+8n_1^3\sum_{v\in V(H)}d_H(v)\\ &+\sum_{uv\in E(H)}(d_H(u)+d_H(v))^4+4n_2\sum_{uv\in E(H)}(d_H(u)+d_H(v))^3\end{aligned}$$

$$
\begin{aligned}
&+6n_2^2 \sum_{uv\in E(H)} (d_H(u)+d_H(v))^2 + 4n_2^3 \sum_{uv\in E(H)} (d_H(u)+d_H(v)) \\
&+n_2^4 \sum_{uv\in E(H)} 1 + n \sum_{v\in V(H_1)} d_{H_1}(v)^4 + 4n \sum_{v\in V(H_1)} d_{H_1}(v)^3 \\
&+6n \sum_{v\in V(H_1)} d_{H_1}(v)^2 + 4n \sum_{v\in V(H_1)} d_{H_1}(v) \\
&+nn_1 + m \sum_{v\in V(H_2)} d_{H_2}(v)^4 + 4m \sum_{v\in V(H_2)} d_{H_2}(v)^3 \\
&+6m \sum_{v\in V(H_2)} d_{H_2}(v)^2 + 4m \sum_{v\in V(H_2)} d_{H_2}(v) + mn_2
\end{aligned}
$$

$$
\begin{aligned}
Y\big(H^T \otimes (H_1^{\cdot}, H_2^{\circ})\big) = {} & 16 \sum_{v\in V(H)} d_H(v)^4 + 32n_1 \sum_{v\in V(H)} d_H(v)^3 \\
&+24n_1^2 \sum_{v\in V(H)} d_H(v)^2 + 8n_1^3 \sum_{v\in V(H)} d_H(v) + nn_1^4 \\
&+ \sum_{uv\in E(H)} (d_H(u)^4 + d_H(v)^4) + 4 \sum_{uv\in E(H)} d_H(u)d_H(v) \\
&[d_H(u)^2 + d_H(v)^2] + 6 \sum_{uv\in E(H)} d_H(u)^2 d_H(v)^2 \\
&+4n_2 \sum_{uv\in E(H)} [d_H(u)^3 + d_H(v)^3] + 12n_2 \sum_{uv\in E(H)} d_H(u)d_H(v) \\
&[d_H(u) + d_H(v)] + 6n_2^2 \sum_{uv\in E(H)} (d_H(u)+d_H(v))^2 \\
&+4n_2^3 \sum_{uv\in E(H)} [d_H(u)+d_H(v)] + n_2^4 \sum_{uv\in E(H)} 1 \\
&+n \sum_{v\in V(H_1)} d_{H_1}(v)^4 + 4n \sum_{v\in V(H_1)} d_{H_1}(v)^3 \\
&+6n \sum_{v\in V(H_1)} d_{H_1}(v)^2 + 4n \sum_{v\in V(H_1)} d_{H_1}(v) + nn_1 \\
&+m \sum_{v\in V(H_2)} d_{H_2}(v)^4 + 4m \sum_{v\in V(H_2)} d_{H_2}(v)^3 \\
&+6m \sum_{v\in V(H_2)} d_{H_2}(v)^2 + 4m \sum_{v\in V(H_2)} d_{H_2}(v) + mn_2.
\end{aligned}
$$

$$
\begin{aligned}
Y\big(H^T \otimes (H_1^{\cdot}, H_2^{\circ})\big) = {} & 16Y(H) + 32n_1F(H) + 24n_1^2M_1(H) + 16mn_1^3 \\
&+nn_1^4 + M_5(H) + 4Z_{3,1}(H) + 6HM_2(H) \\
&+4n_2Y(H) + 12n_2ZeZM(H) + 6n_2^2HM_1(H) \\
&+4n_2^3M_1(H) + mn_2^4 + nY(H_1) + 4nF(H_1) \\
&+6nM_1(H_1) + 8m_1n + nn_1 + mY(H_2) \\
&+4mF(H_2) + 6mM_1(H_2) + 8mm_2 + mn_2
\end{aligned}
$$

□

Example 3.12. Using the above equation by $Y\left(H^T \otimes (H_1^{\cdot}, H_2^{\circ})\right)$,

$$\begin{aligned}
Y[P_l^T \otimes (C_m^{\cdot}, C_n^{\circ})] &= n^4(l-1) + 8n^3(2l-3) \\
&+12n^2(8l-15) + lm^4 \\
&+16m^3(l-1) + 12m^2(8l-15) \\
&+337l(m+n) - 633n \\
&-448m + 512l - 1086 && ,l \geq 2, n, m \geq 3 \\
Y[C_l^T \otimes (P_m^{\cdot}, C_n^{\circ})] &= l(m^4+n^4) + 16l(m^3+n^3) \\
&+96l(m^2+n^2) + l(337m+145n) \\
&+382l && ,l, m, n \geq 3 \\
Y[P_l^T \otimes (P_m^{\cdot}, P_n^{\circ})] &= l(m^4+n^4) + 16l(m^3+n^3) \\
&+96l(m^2+n^2) + 337lm + 81ln \\
&+156l - n^4 - 24n^3 \\
&-240n^2 - 81n - 16m^3 \\
&-144m^2 - 448m - 716 && ,l, n, m \geq 2 \\
Y[C_l^T \otimes (C_m^{\cdot}, C_n^{\circ})] &= l(m^4+n^4) + 16l(m^3+n^3) \\
&+96l(m^2+n^2) + 337l(m+n) \\
&+412l - 62 && ,l, m, n \geq 3
\end{aligned}$$

3.5. Application. PAMAM Dendeimer (Highly branched Polymer)

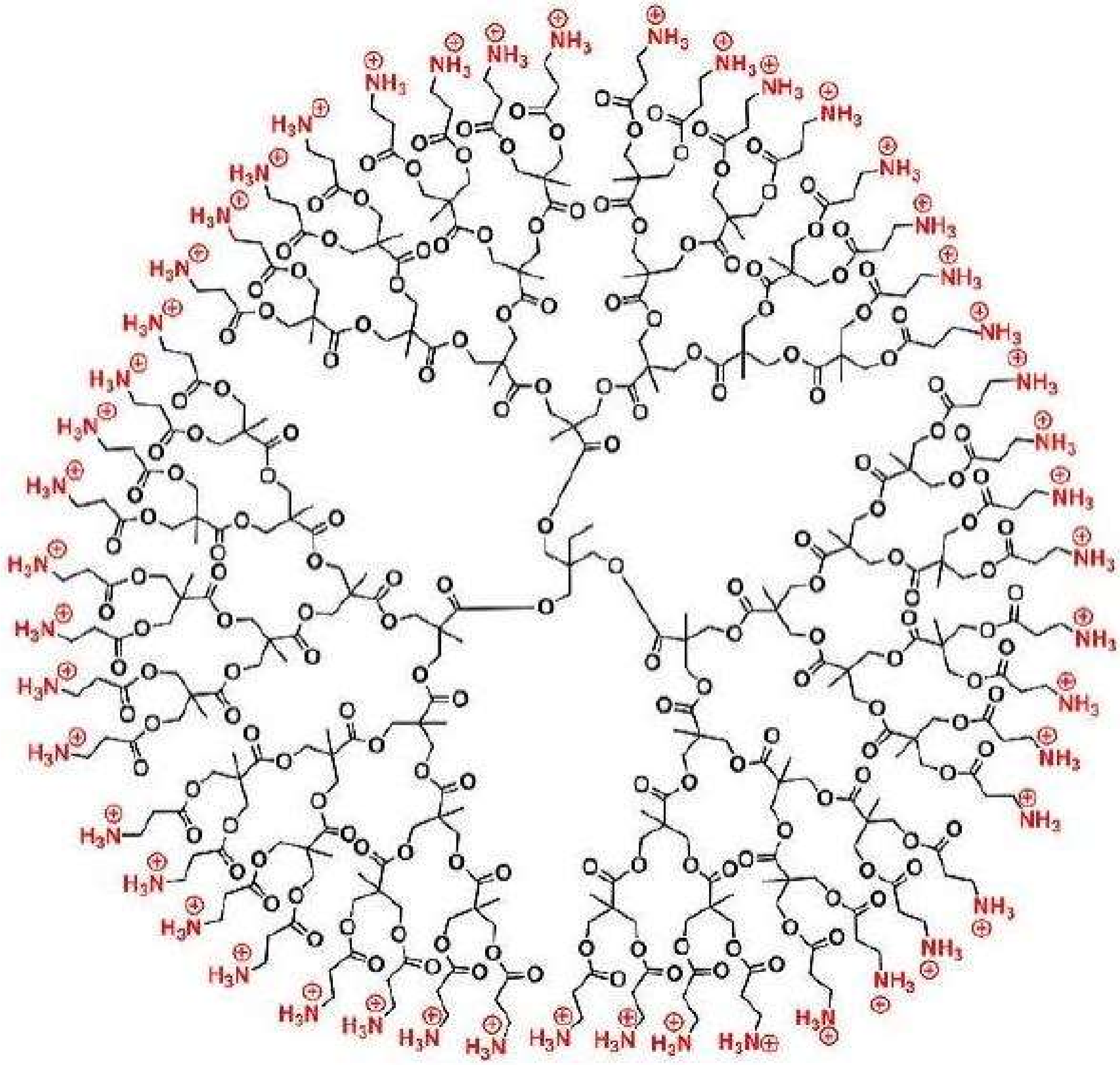

Figure 5

**Degree Classification**

| Vertices Based | Chemical Bonding | Degree | Count |
|---|---|---|---|
| Core | Ethylenediamine | 2 | 1 |
| Tertiary amine | branching N atoms | 3 | $N_3(H)$ |
| Amicle Carbon | linkage atoms | 2 | $N_2(H)$ |
| Terminal amine | $NH_2$ $groups$ | 1 | $N_1(H)$ |

**Structure of Vertices:**

Terminal groups : $N_1(H) = 2^{H+1}$

Branching(tertiaryamines) : $N_3(H) = 2^H - 1$

Linkage(amideunits) : $N_2(H) = 2^{H+1} - 2$

**Y index**

$$\begin{aligned}
Y(H) &= \sum d(v)^4 \\
&= N_1(H) + 1^4 + N_2(H) \cdot 2^4 + N_3(H) \cdot 3^4 + \text{ core.} \\
Y(H) &= 2^{H+1} + 16(2^{H+1} - 2) + 81(2^H - 1) + 16 \\
&= 2^{H+1} + 16(2^{H+1}) - 16(2) + 81(2^H) - 81 + 16 \\
&= 2^{H+1}(17) + 81(2^H) - 97 \\
&= (2(17) + 81)2^H - 97 \\
Y(H) &\simeq 115 \cdot 2^H - 97
\end{aligned}$$

**3.6. Numerical Comparision.**

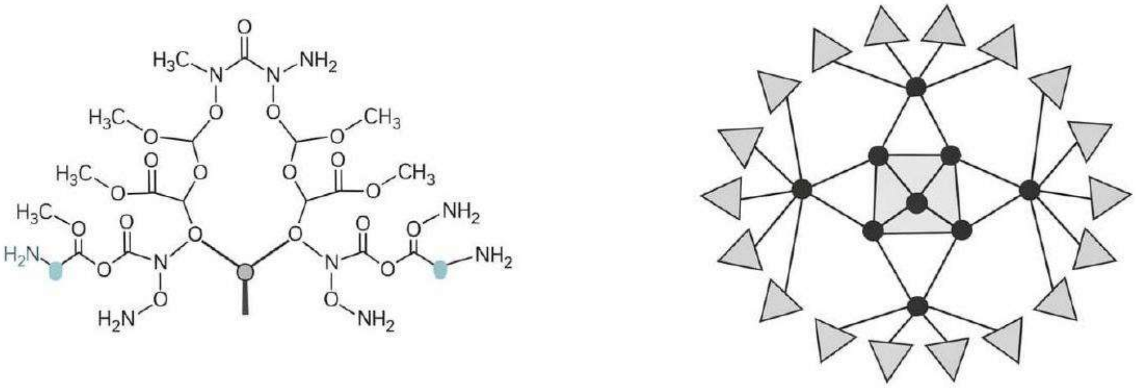

Figure 6

| Graph | PAMAM Y(H) | Double Corona graph Y(H) |
|---|---|---|
| H | 133 | ≃200 |
| $H_1$ | 363 | ≃400 |
| $H_2$ | 823 | ≃800 |

A numerical comparison for $PAMAM$ dendrimers and double corona graphs exhibit exponential growth in the $Y$-topological index.

Although the exact values differ due to variations in degree distribution, the overall growth pattern remains comparable, confirming that double corona graphs provide a suitable mathematical model for dendritic molecular structure. The closeness of values for higher generations indicates asymptotic similarity between the two models.

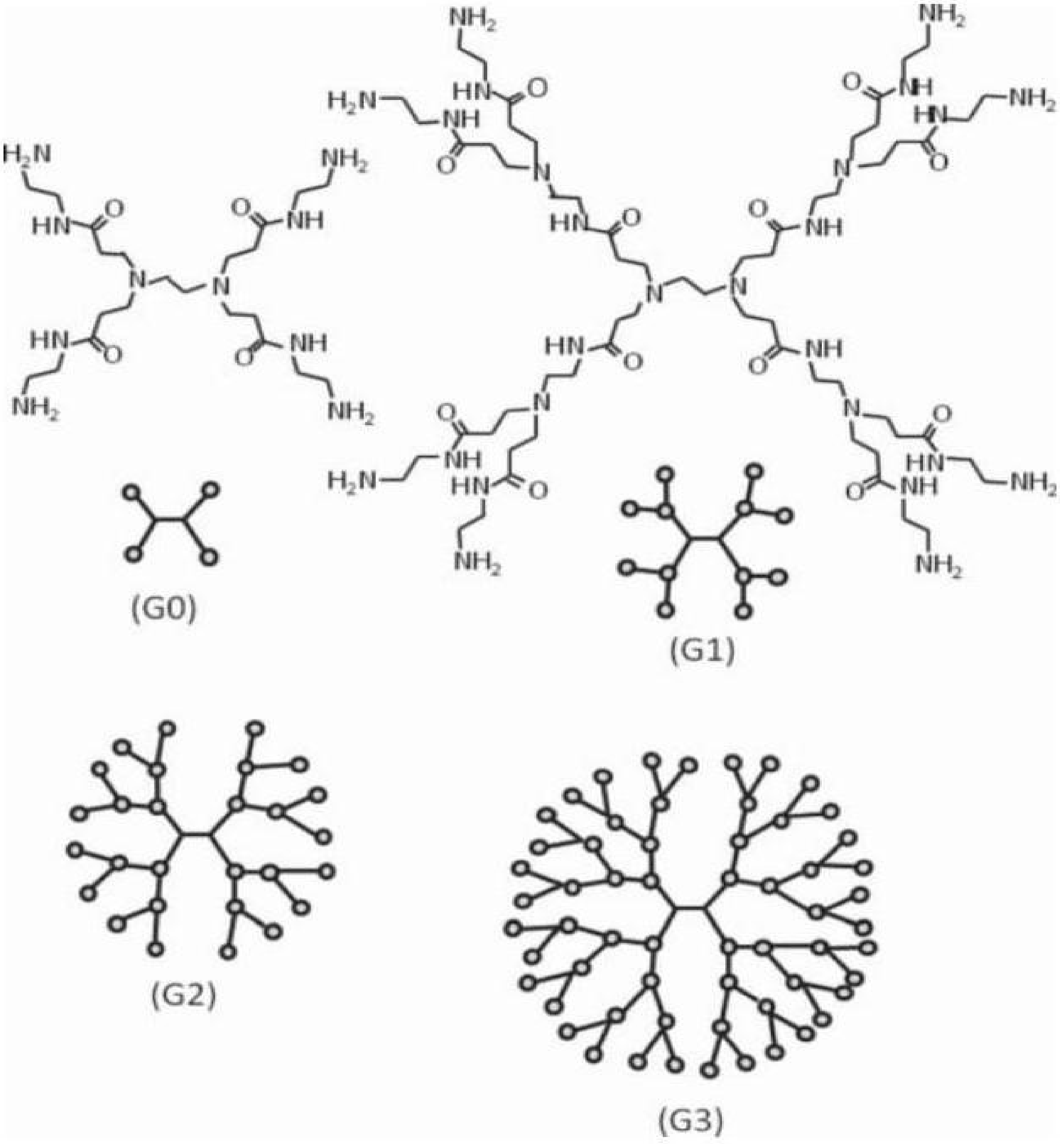

Figure 7

## 4. Conclusion

In this study, we derived explicit closed form for $Y$-topological index of double corona graphs related to subdivision operations. A structural analysis between double corona graph and PAMAM dendrimers is validated and discussed their exponential growth of the $Y$-index. We demonstrate the double corona graph constructions as mathematical models for dentritic molecular as structures and their potential applications in chemical graph theory and QSPR analysis

## References

[1] Alameri, A., Al-Naggar, N., Al-Rumaima, M., & Alsharafi, M. (2020). Y-index of some graph operations. International Journal of Applied Engineering Research (IJAER), 15(2), 173-179.

[2] BERHE, B. M., & Wang, C. (2020). The first general Zagreb coindex of graph operations. Applied Mathematics and Nonlinear Sciences, 5(2), 109-120.

[3] Cheng, S., & Wu, B. (2024). The first general Zagreb index of graphs and their line graphs. Journal of Applied Mathematics and Computing, 70(3), 1937-1951.

[4] De, N., Sarkar, P., & Pal, A. (2019). The Forgotten Topological Index of Double Corona of Graphs Related

to the Different Subdivision Graphs. Mathematical Advances in Pure and Applied Sciences, 2(1), 8-15.

[5] Furtula, B., & Gutman, I. (2015). A forgotten topological index. Journal of mathematical chemistry, 53(4), 1184-1190. and Its Application. 2013;438(8):3547-3559.

[6] Gao, W., Farahani, M. R., Jamil, M. K., & Siddiqui, M. K. (2016). The redefined first, second and third Zagreb indices of titania nanotubes TiO2 $[m, n]$. The Open Biotechnology Journal, 10(1).

[7] I Gutman, N Trinajestic, Graph theory and molecular orbitals total $\pi$ - electron energy of alternant hydrocarbons, Chemical Physics Letters, 1972(17), 535-538.

[8] Ivan Gutman, On degree and distance based topological indices, Academia Romana- Rev, Roum. Chim, 2021, 66(2), 119-123.

[9] Haseeb Ahmad, Muhammad Kamran Siddiqui, Muhammad Farhan Hanif, Brima Gegbe, Bilal M, Waheed M., Exploring the Distance-Based Topological Indices for Total Graphs via Numerical Comparison, Journal of Mathematics, 2024, 19 pages, DOI: 10.1155/jom/4423113

[10] Hua, H., Das, K. C., & Wang, H. (2019). On atom-bond connectivity index of graphs. Journal of Mathematical Analysis and Applications, 479(1), 1099-1114.

[11] Khalifeh, M. H., Yousefi-Azari, H., & Ashrafi, A. R. (2009). The first and second Zagreb indices of some graph operations. Discrete applied mathematics, 157(4), 804-811.

[12] Li, R. (2025). The first Zagreb index, the independence number and some Hamiltonian properties of graphs. Open Journal of Discrete Mathematics, 15, 92-99.

[13] Li, X., & Zheng, J. (2005). A unified approach to the extremal trees for different indices. MATCH Commun. Math. Comput. Chem, 54(1), 195-208.

[14] Maji, D., & Ghorai, G. (2020). Computing F-index, coindex and Zagreb polynomials of the kth generalized transformation graphs. Heliyon, 6(12).

[15] Nagarajan, S., Kayalvizhi, G., & Priyadharsini, G. (2021). S-index of different graph operations. Asian Research Journal of Mathematics, 17(12), 43-52.

[16] Rajasekharaiah, G. V., & Murthy, U. P. (2021). Hyper-Zagreb indices of graphs and its applications. Journal of Algebra Combinatorics Discrete Structures and Applications, 8(1), 9-22.

[17] Robin J. Wilson, Introduction to Graph Theory (2010) - 4th edition, Longman Group Ltd 1979, 1-184.

[18] Sirous Ghobadi, Mobina Ghorbaninejad, On F-Polynomial, Multiple and Hyper F-Index of some Molecular Graphs, Bulletin of Mathematical Sciences and Applications, 2018, 20:36-43, DOI: 10.18052/www.scipress.com/BMSA.20.36

[19] Vujošević, S., Popivoda, G., Vukićević, Ž. K., Furtula, B., & Škrekovski, R. (2021). Arith-metic-geometric index and its relations with geometric-arithmetic index. Applied Mathematics and Computation, 391, 125706.

[20] Wiener, H. (1947). Structural determination of paraffin boiling points. Journal of the American chemical society, 69(1), 17-20.

[21] Yuan, Y., Zhou, B., & Trinajstić, N. (2010). On geometric-arithmetic index. Journal of mathematical chemistry, 47(2), 833-841.