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ANALYTIC ODD MEAN LABELING OF SOME DUPLICATE GRAPHS

S.Sriram
Associate Professor, P.G Department of Mathematics, Patrician College of Arts and Science, Adyar, Chennai-20, India Email: sanksriram@gmail.com

A.Selvaganapathy
Assistant Professor, Department of Mathematics , S R M College of Arts and Science, Kattangulathur, Chennai, India

K.Thirusangu
Principal & Head(Retd.), P.G. Department of Mathematics, SIVET College , Gowrivakkam, Chennai, India .

Abstract: Let $G = (V, E)$ be a finite and simple graph with p vertices and q edges. In this graph we discuss on the analytic odd mean labeling of some duplicate graphs. A graph is said to be an analytic odd mean labeling graph if there exists an injective function $\kappa: V \to \{0,1,3, \dots ,2q - 1\}$ for which the induced edge labeling $\kappa^*: E \to Z$ such that for each edge ab with $\kappa(a) < \kappa(b)$ such that

$$\kappa^*(ab) = \begin{cases} \left\lceil \frac{\kappa(b)^2-(\kappa(a)+1)^2}{2} \right\rceil & if\ \kappa(a) \neq 0 \\ \left\lceil \frac{\kappa(b)^2}{2} \right\rceil & if \quad \kappa(a) = 0 \end{cases}$$

is injective. In this paper we have examined on duplicate of Path , Cycle, Wheel , Helm and Comb graph and have proved that these graphs are analytic odd mean labeling graphs.

Key Words: Analytic odd mean labeling, Duplicate graph, Path, Cycle, Wheel , Helm and Comb graph
AMS Subject Classification (2010) : 05C78

1. Introduction

In this paper we consider only finite and simple graph G with p vertices and q edges. In the study of graph labeling techniques innumerous labeling techniques were introduced and many graphs have been proved to hold the labeling conditions and have been compiled by J.A.Gallian[1]. The graph labeling concepts were initially introduced by Rosa and later the cordial labeling techniques was introduced by Cahit . Many more researchers have established on these ideas and carried out their research in various labeling. Analytic Odd Mean labeling was introduced by P.Jeyanthi et.al [2,3] and have proved the path, cycle, Complete graph, Wheel graph, complete bipartite graph and Flower graph are analytic odd mean labeling graphs. Further in their research they proved many other graphs are analytic odd mean labeling. Motivated towards the labeling technique we further studied various papers on duplicate graphs [4,5,6] and we in this paper have proved duplicate of Path, cycle, wheel , Helm and Comb graph to be analytic odd mean labeling graphs.

2. Preliminaries

Definition .2.1 : **Analytic Odd Mean Labeling Graph**

A graph G is said to be analytic odd mean labeling graph if there exists and injective function $\kappa: V \rightarrow \{0,1,3,\ldots,2q-1\}$ for which the induced edge labeling $\kappa^*: E \rightarrow Z$ such that for each edge ab with $\kappa(a) < \kappa(b)$ such that $\kappa^*(ab) =$

$$\begin{cases} \left\lceil \frac{\kappa(b)^2 - (\kappa(a)+1)^2}{2} \right\rceil & if\ \kappa(a) \neq 0 \\ \left\lceil \frac{\kappa(b)^2}{2} \right\rceil & if \quad \kappa(a) = 0 \end{cases}$$ is injective.

Definition. 2.2 : **Duplicate Graphs**

For a graph $G = (V, E)$ with vertex set and edge set the duplicate graph is defined as $DG = (V_1, E_1)$ where the vertex set $V_1 = V \cup V'$ and $V \cap V' = \phi$ and $h: V \rightarrow V'$ is bijective. The edges set E_1 of DG is defined as a edge $ab \in E$ if and only if both edges $ab', a'b$ are in E_1.

Definition.2.3 : **Path graph**

A linear sequence of n vertices where edges connect consecutive vertices is called the path graph P_n.

Definition.2.4 : **Cycle graph**

A closed chain of connected graph is called a cycle graph C_n.

Definition.2.4 : **Wheel graph**

A wheel graph $W_n, n \geq 3,$ is obtained by joining the vertices of a cycle graph C_n to a common vertex called the centre. The vertices of degree 3 are called the rim vertices.

Definition.2.5 : **Helm graph**

A Helm graph H_n is the graph obtained from an n wheel graph by adjoining a pendant edge at each node of the cycle.

Definition.2.6 : **Comb graph**

A comb graph $P_n \circ K_1$ is a graph in which at each of the vertex of the path graph P_n a pendant edge is connected. The comb graph consists of 2n vertices and 2n-1 edges.

3. **Main Results**.

Theorem.3.1 : The Duplicate of Path graph $DG(P_n)$ is analytic odd mean labeling graph

Proof: Consider the duplicate of path graph $DG(P_n)$. Let the vertex set $V(G) = \{a_1, a_2, a_3, \ldots, a_n, a_1', a_2', a_3', \ldots, a_n'\}$ and edge set $E(G) = \{(a_i a_{i+1}') \cup (a_i' a_{i+1}), 1 \leq i \leq n-1\}$_

Therefore, the duplicate of path graph $DG(P_n)$ consists of 2n vertices and 2n-2 edges .

Let us define an injective function $\kappa: V \rightarrow \{0,1,3,\ldots,2q-1\}$. Let us label the vertices as follows

$\kappa(a_1) = 0$

$\kappa(a_{2i}) = 8i - 3$ for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$k(a_{2i+1}) = 8i + 1$ for $1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$

$\kappa(a_1') = 3,\ \kappa(a_2') = 1,$

$\kappa(a'_{2i+1}) = 8i - 1$ for $1 \leq i \leq \left\lfloor\frac{n-1}{2}\right\rfloor$

$\kappa(a'_{2i}) = 8i + 3$ for $1 \leq i \leq \left\lfloor\frac{n-2}{2}\right\rfloor$

Then the induced edge labeling is given as follows

$\kappa(a_1 a'_2) = 1$

$\kappa(a_{2i} a'_{2i+1}) = 8i - 1$, for $1 \leq i \leq \left\lfloor\frac{n-1}{2}\right\rfloor$

$\kappa(a'_{2i+1}\, a_{2i+2}) = 8i + 3$ for $1 \leq i \leq \left\lfloor\frac{n-1}{2}\right\rfloor$,

$\kappa(a'_1 a_2) = 5$, $\kappa(a'_2 a_3) = 39$

$\kappa(a'_{2i+1}\, a_{2i+2}) = 20i + 33$ for $1 \leq i \leq \frac{n-1}{2}$, where n is odd

We find the above induced edge labeling satisfies the condition of analytic odd mean labeling and

hence the duplicate of path graph $DG(P_n)$ is analytic odd mean labeling graph. Hence the proof.

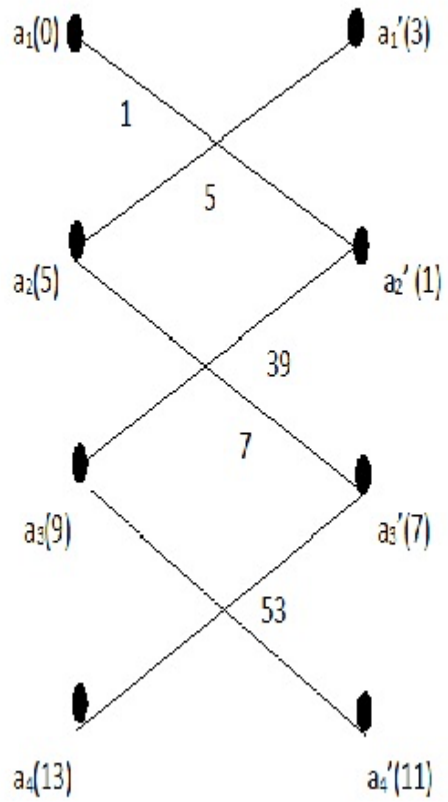

Fig..3.1.1 Duplicate of Path graph $DG(P_4)$

Theorem.3.2 : The Duplicate of Cycle graph $DG(C_n)$ is analytic odd mean labeling graph.

Proof: Consider the duplicate of cycle graph $DG(C_n)$. Let the vertex set

$V(G) = \{a_1, a_2, a_3, \ldots, a_n, a'_1, a'_2, a'_3, \ldots, a'_n\}$ and the edge set
$E(G) = \{(a_i a'_{i+1}) \cup (a'_{i+1}\, a_i) \cup (a_n a'_1) \cup (a_1 a'_n), 1 \le i \le n-1\}$.

The duplicate of cycle graph $DG(C_n)$

consists of 2n vertices and 2n edges. Let us define an injective function $\kappa: V \to \{0,1,3, \ldots, 2q-1\}$.

Let us label the vertices as follows

$\kappa(a_1) = 0$

$\kappa(a_{2i}) = 8i - 3$ for $1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$

$k(a_{2i+1}) = 8i + 1$ for $1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$,

$\kappa(a'_1) = 3,\ \kappa(a'_2) = 1$,

$\kappa(a'_{2i+1}) = 8i - 1$ for $1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$,

$\kappa(a'_{2i}) = 8i - 5$ for $2 \le i \le \left\lfloor \frac{n-2}{2} \right\rfloor$,

Then the induced edge labeling is obtained as follows

$\kappa(a_1 a'_2) = 1$

$\kappa(a_{2i} a'_{2i+1}) = 8i - 1$, for $1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$

$\kappa(a_{2i+1} a'_{2i+2}) = 8i + 3$ for$1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$

$\kappa(a'_1 a_2) = 5$, $\kappa(a'_2 a_3) = 39$

$\kappa(a'_{2i+1}\, a_{2i+2}) = 20i + 33$ for $1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$

$\kappa(a_n a'_1) = \left\lceil \frac{(4n-3))^2 - 16}{2} \right\rceil$.

$\kappa(a_1 a'_n) = \left\lceil \frac{(4n-5)^2}{2} \right\rceil$

We find the above induced edge labeling satisfies the condition of analytic odd mean labeling and

hence the duplicate of cycle graph $DG(C_n)$ is analytic odd mean labeling graph. Hence the proof.

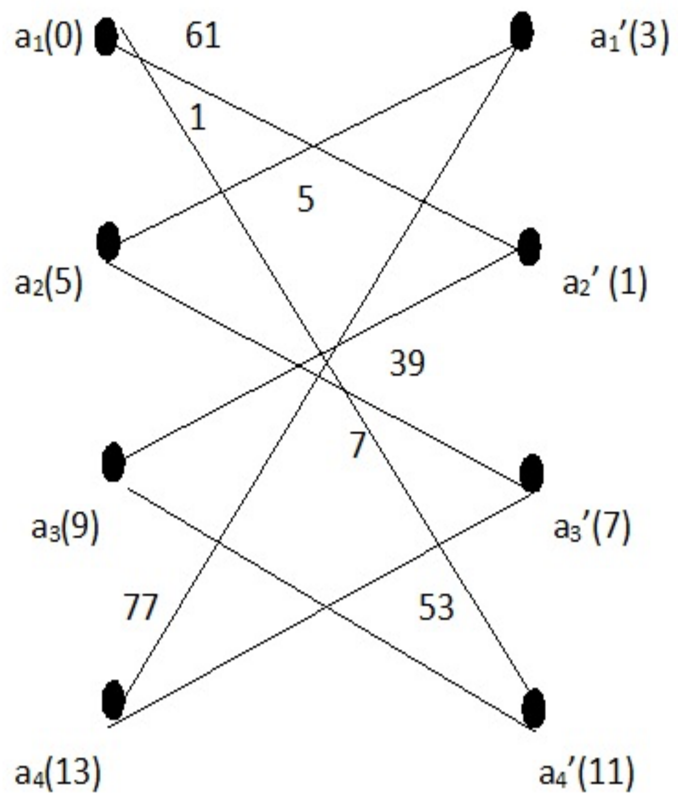

Fig.3.2.1 Duplicate of Cycle graph $DG(C_4)$

Theorem.3.3: The Duplicate of wheel graph $DG(W_n)$ is analytic odd mean labeling graph.

Proof: Consider the duplicate of wheel graph $DG(W_n)$ with vertex set

$V(G) = \{a_0, a_1, a_2, a_3, \ldots, a_n, a_0', a_1', a_2', a_3', \ldots, a_n'\}$ and edge set

$E(G) = \{(a_i a_{i+1}') \cup (a_{i+1}' a_i) \cup (a_0 a_i') \cup (a_0' a_i), 1 \leq i \leq n-1\}$_ Where the vertex a_0 is the

centre of the wheel and a_0' is the duplicate of the centre vertex a_0. Therefore the number of vertices

of duplicate wheel graph $DG(W_n)$ is $2n+2$ and the number of edges is 4n-2 . Let us define an

injective function $\kappa: V \to \{0,1,3,\ldots,2q-1\}$. Now let us label the vertices as follows

$\kappa(a_0) = 0$; $\kappa(a_0') = 1$

$\kappa(a_1) = 3$; $\kappa(a_1') = 7$; $\kappa(a_2') = 5$

$\kappa(a_{2i}) = 8i + 1, \; 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$,

$\kappa(a_{2i+1)} = 8i + 5, 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.,

$\kappa(a_{2i+1}') = 8i + 3, \; 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$,

$\kappa(a_{2i+2}') = 8i + 7, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

Then we can find the induced edges as follows

$\kappa(a_1 a_2') = 5$; $\kappa(a_{2i} a'_{2i+1}) = 8i + 3$, $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a_{2i+1}\, a'_{2i+2}) = 8i + 7.$

$\kappa(a_1' a_2) = 9$; $\kappa(a_2' a_3) = 67;$

$\kappa(a'_{2i+1}\, a_{2I+2}) = 40i + 33$, $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a'_{2i+2}\, a_{2i+3}) = 40i + 53.$

$\kappa(a_0 a_1') = 25$;

$\kappa(a_0 a_2') = 13$;

$\kappa(a_0 a'_{2i+1}) = \left\lceil \frac{(8i+3)^2}{2} \right\rceil$, $1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$

$\kappa(a_0 a'_{2i+2}) = \left\lceil \frac{(8i+7)^2}{2} \right\rceil$, $1 \leq i \leq \frac{n}{2}$

$\kappa(a_0', a_1) = 3$;

$\kappa(a_0', a_{2i}) = \left\lceil \frac{(8i+1)^2-4}{2} \right\rceil$, $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a_0', a_{2i+1}) = \left\lceil \frac{(8i+5)^2-4}{2} \right\rceil$, $1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

$\kappa(a_n a_1') = \left\lceil \frac{(4n+1))^2-49}{2} \right\rceil$.

$\kappa(a_1 a_n') = \left\lceil \frac{(4n-1)^2-9}{2} \right\rceil$

We find the above induced edge labeling satisfies the condition of analytic odd mean labeling and

hence the duplicate of wheel graph $DG(W_n)$ is analytic odd mean labeling graph. Hence the proof.

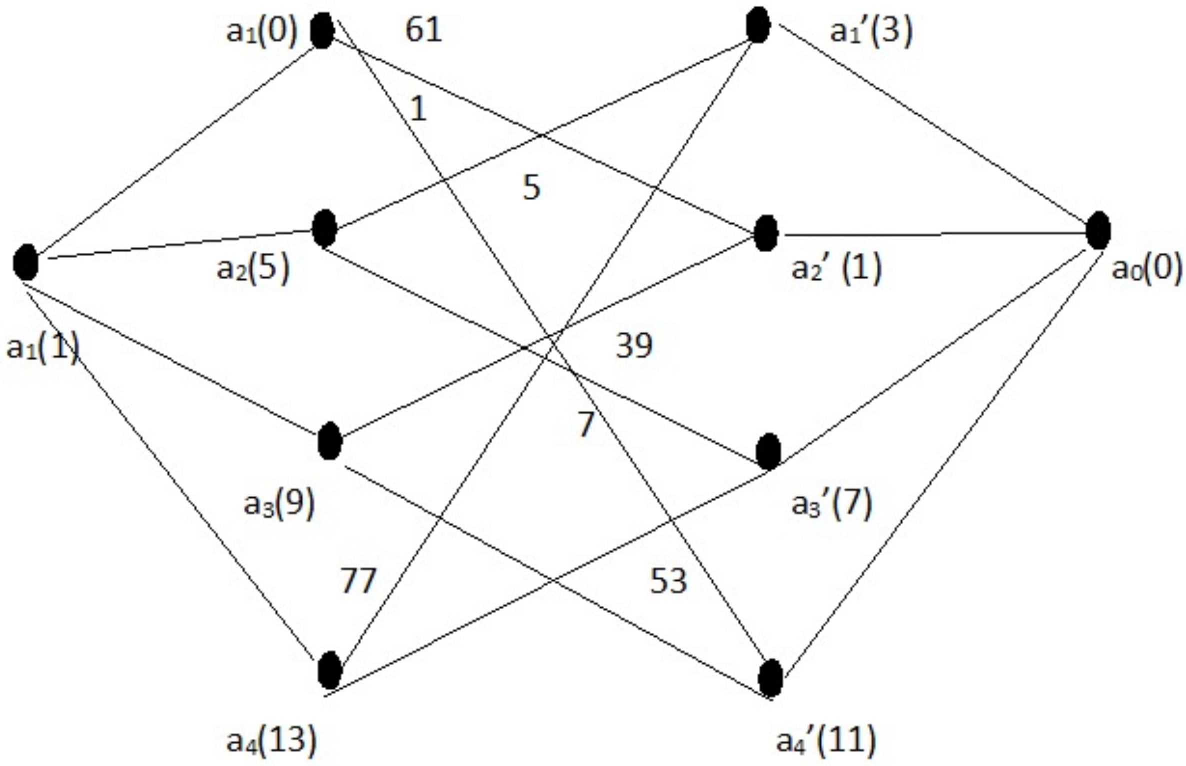

Fig.3.3.1 Duplicate of Wheel graph $DG(W_4)$

Theorem.3.4 : The duplicate of Helm graph $DG(H_n)$ is analytic odd mean labeling graph.

Proof: Consider the duplicate of Helm graph G= $DG(H_n)$. The vertex set is $V(G) = \{a_1, a_2, \dots, a_n, a'_1, a'_2, \dots, a'_n, a_0, a'_0, b_1, b_2, \dots, b_m, b'_1, b'_2, \dots, b'_m\}$ and the edge set is $E(G) = \{(a_i a'_{i+1}) \cup (a'_i a_{i+1}) \cup (a_0 a'_i) \cup (a'_0 a_i) \cup (a_i b'_i) \cup (a'_i b_i)$. Therefore the number of vertices of duplicate of Helm graph is 2n+2m+2 and the number of edges is 6n .

Let us define an injective function $\kappa: V \to \{0,1,3, \dots, 2q-1\}$. Now let us label the vertices as follows

$\kappa(a_0) = 0$; $\kappa(a'_0) = 1$

$\kappa(a_1) = 3$; $\kappa(a'_1) = 7$; $\kappa(a'_2) = 5$

$\kappa(a_{2i}) = 8i + 1$, $1 \le i \le \left\lfloor\frac{n}{2}\right\rfloor$

$\kappa(a_{2i+1}) = 8i + 5$, $1 \le i \le \left\lfloor\frac{n-1}{2}\right\rfloor$.

$\kappa(a'_{2i+1}) = 8i + 3$, $1 \le i \le \left\lfloor\frac{n-1}{2}\right\rfloor$

$\kappa(a'_{2i+2}) = 8i + 7$, $1 \le i \le \left\lfloor\frac{n}{2}\right\rfloor$

$\kappa(b'_j) = (4n + 4j - 1)$, $1 \le j \le \left\lfloor\frac{m}{2}\right\rfloor$.

$\kappa(b_j) = (4n + 4j + 1)$, $1 \le j \le \left\lfloor\frac{m}{2}\right\rfloor$

Then the induced edge labeling is obtained as follows

$\kappa(a_1 a_2') = 5$; $\kappa(a_{2i} a_{2i+1}) = 8i + 3$, $1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$

$\kappa(a_{2i+1}\, a_{2i+2}') = 8i + 7.\ 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a_1' a_2) = 9$; $\kappa(a_2' a_3) = 67$;

$\kappa(a_{2i+1}'\, a_{2I+2}) = 40i + 33, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a_{2i+2}'\, a_{2i+3}) = 40i + 53, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a_0 a_1') = 25$;

$\kappa(a_0 a_2') = 13$;

$\kappa(a_0 a_{2i+1}') = \left\lceil \frac{(8i+3)^2}{2} \right\rceil,\ 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a_0 a_{2i+2}') = \left\lceil \frac{(8i+7)^2}{2} \right\rceil,\ 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a_0', a_1) = 3$;

$\kappa(a_0', a_{2i}) = \left\lceil \frac{(8i+1)^2-4}{2} \right\rceil,\ \ 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a_0', a_{2i+1}) = \left\lceil \frac{(8i+5)^2-4}{2} \right\rceil, 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

$\kappa(b_1' a_1) = 105$;

$\kappa(b_j' a_{2i}) = \left\lceil \frac{(4n+4j-1)^2-(8i+2)^2}{2} \right\rceil, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor,\ 2 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor$

$\kappa(b_j' a_{2i+1}) = \left\lceil \frac{(4n+4j-1)^2-(8i+6)^2}{2} \right\rceil, 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor,\ 2 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor$

$\kappa(b_1 a_1') = 113$; $\kappa(b_2 a_2') = 203$

$\kappa(b_j a'_{2i+1}) = \left\lceil \frac{(4n+4j+1)^2-(8i+4)^2}{2} \right\rceil, 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor,\ 3 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor$.

$\kappa(b_j a'_{2i+2}) = \left\lceil \frac{(4n+4j+1)^2-(8i+8)^2}{2} \right\rceil, 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor,\ 3 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor$.

$\kappa(a_n a_1') = \left\lceil \frac{(4n+1))^2-49}{2} \right\rceil.\quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a_1 a_n') = \left\lceil \frac{(4n-1)^2-9}{2} \right\rceil, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

We find the above induced edge labeling satisfies the condition of analytic odd mean labeling and

hence the duplicate of helm graph $DG(H_n)$ is analytic odd mean labeling graph. Hence the proof.

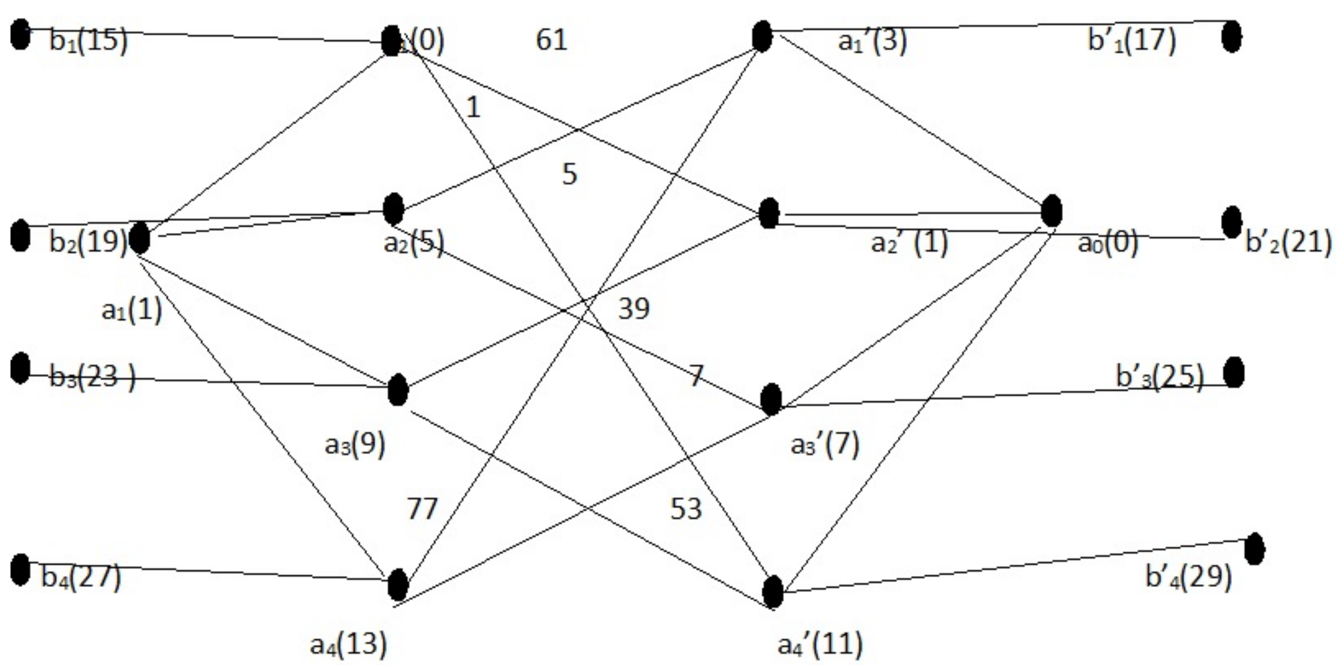

Fig.3.4.1 Duplicate of Helm graph $DG(H_4)$

Theorem.3.5 : The duplicate of comb graph $DG(P_n \circ K_1)$ is analytic odd mean labeling graphs.

Proof: Consider the duplicate of comb graph $G = DG(P_n \circ K_1)$. The vertex set is $V(G) = \{a_1, \dots, a_n, a'_1, a'_2, \dots, a'_n, b_1, b_2, b_3, \dots, b_m b'_1, b'_2, \dots, b'_m\}$ and the edge set is $E(G) = \{(a_i a'_{i+1}) \cup (a'_i a_{i+1}) \cup (a_i b') \cup (a'_i b_i)$. The number of vertices of duplicate of comb graph is 2n+2m. The number of edges of duplicate of comb graph is 2n+6 . Let us define an injective function $\kappa: V \to \{0,1,3, \dots ,2q-1\}$. Now let us label the vertices as follows

$\kappa(a_1) = 0$

$\kappa(a_{2i}) = 8i - 3 \ , \ 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$

$k(a_{2i+1}) = 8i + 1 \ , \ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a'_1) = 3, \ \kappa(a'_2) = 1,$

$\kappa(a'_{2i+1}) = 8i - 1 \, , 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a'_{2i}) = 8i + 3 \, , 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$

$\kappa(b'_j) = (4n + 4j - 5) \ \ 1 \le j \le m$

$\kappa(b_j) = (4n + 4j - 3) \quad 1 \le j \le m$

Then the induced edge labeling is given as follows

$\kappa(a_1 a'_2) = 1$

$\kappa(a_{2i} a'_{2i+1}) = 8i - 1 \, , 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a'_{2i+1}\, a_{2i+2}) = 8i + 3 \;\; 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$

. $\kappa(a'_1 a_2) = 5$, $\kappa(a'_2 a_3) = 39$

$\kappa(a'_{2i+1}\, a_{2i+2}) = 20i + 33 \quad , 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a_1 b'_1) = \left\lceil \frac{(4n+4-5)^2}{2} \right\rceil$

$\kappa\left(a_{2i} b'_j\right) = \left\lceil \frac{(4n+4j-5)^2-(8i-3)^2}{2} \right\rceil, 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$

$\kappa\left(a_{2i+1} b'_j\right) = \left\lceil \frac{(4n+4j-5)^2-(8i+1)^2}{2} \right\rceil, 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$

$\kappa(a'_1 b_1) = \left\lceil \frac{(4n+1)^2-9}{2} \right\rceil$

$\kappa(a'_2 b_2) = \left\lceil \frac{(4n+5)^2-1}{2} \right\rceil$

$\kappa\left(a'_{2i+1}\, b_j\right) = \left\lceil \frac{(4n+4j-3)^2-(\,8i-1\,)^2}{2} \right\rceil, 1 \le i \le \frac{n-1}{2}\,,\; 3 \le j \le m$

$\kappa\left(a'_{2i}\, b_j\right) = \left\lceil \frac{(4n+4j-3)^2-(\,8i+3\,)^2}{2} \right\rceil, 1 \le i \le \frac{n-1}{2}\,,\; 3 \le j \le m.$

We find the above induced edge labeling satisfies the condition of analytic odd mean labeling and

hence the duplicate of path graph $DG(P_n)$ is analytic odd mean labeling graph. Hence the proof.

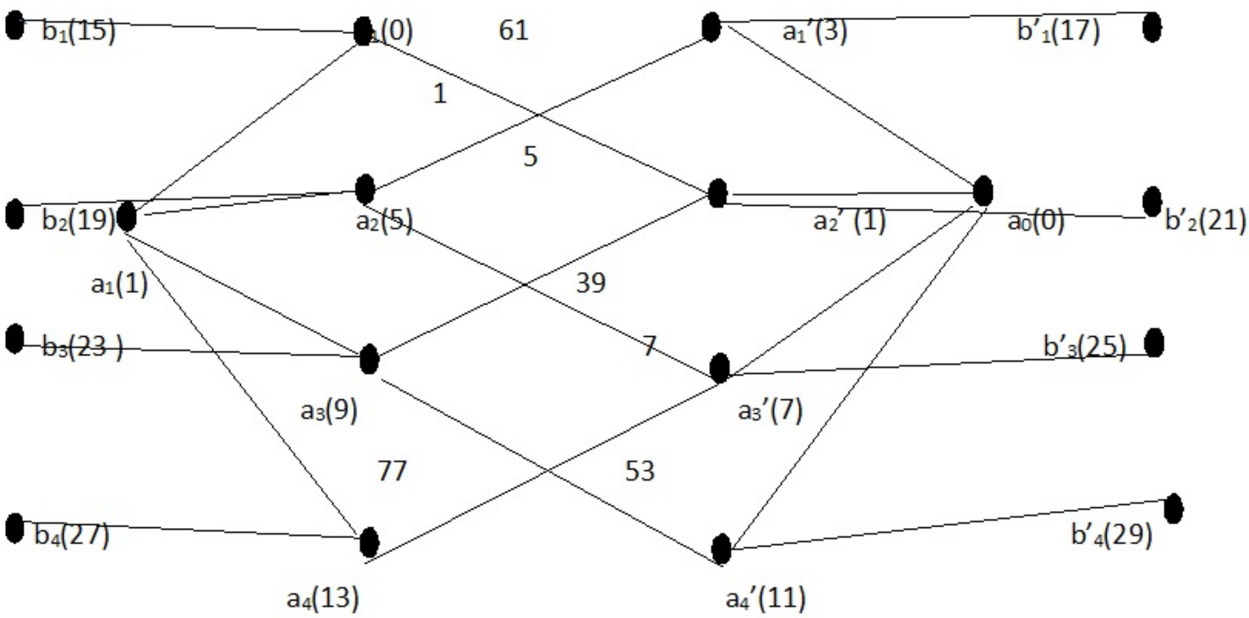

Fig.3.5.1: Duplicate of Comb graph $DG(P_n \circ K_1)$

4. **Conclusion**:

In this paper we have identified duplicate graphs of Path, Cycle, Wheel, Helm and Comb and have proved that these graphs satisfies the condition of analytic odd mean labeling. We further in our future wish to study on some classes of graphs and prove that they obey analytic odd mean labeling.

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