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UNIQUE STRONG ISOLATE SUPER DOMINATION IN GRAPHS

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Abstract

A dominating set D of $V(G)$ in a graph $G=(V,E)$ is called super dominating set if for every $v \in V(G)-D$, there exists an external private neighbour of v with respect to $V(G)-D$. A super dominating set D of a graph G is said to be an Isolate Super Dominating Set (ISD-set) of G if $\langle D \rangle$ has at least one isolated vertex. An ISD-set D is considered as the Unique Strong Isolate Super Dominating Set (USISD-set), if there exists exactly one isolated vertex $a \in D$ such that $N_2(a) \cap D = \emptyset$, where $N_2(a) = \{ b : d(a,b) \leq 2 \text{ and } a \neq b \}$. The unique strong isolate super domination number (USISD-number), denoted by $\gamma^{U,S}_{0,sp}(G)$, is the minimum cardinality of a the unique strong isolate super dominating set of G. In this paper, we initiate a study on this parameter. We obtain basic properties of the unique strong isolate super dominating sets in graphs. Also we present upper and lower bounds for the unique strong isolate super domination number.

Keywords: Super domination, Isolate super domination, Strongly isolate super domination, Unique strong isolate super domination.
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1. Introduction

Domination is a well-established area of graph theory that focuses on the study of vertex subsets capable of exerting control over the entire graph through adjacency. A set $D \subseteq V(G)$ is called a dominating set of a graph G if every vertex not in D is adjacent to at least one vertex of D. The minimum cardinality of such a set is known as the domination number and is denoted by $\gamma(G)$. Since its introduction, domination has been widely investigated due to its strong connections with both theoretical and applied problems, including network monitoring, resource placement, communication systems and social network analysis. Two such notable concepts are Super domination which was formally introduced in 2015 by Lemańska, Swaminathan, Venkatakrishnan, and Zuazua [3], and Isolate domination in 2016 by I.Sahul Hamid and S.Balamurugan [4]. Those concepts have garnered considerable interest due to their abilities to highlights specialized types of dominating sets that are structurally robust and possess interesting mathematical properties. For $X \subseteq V(G)$ and $x \in X$, the set $PN(x,X) = N_G[x] - N_G[X - \{x\}]$ is called the private neighbourhood of x with respect to X [3]. An element u of $PN(x, X)$ is called a private neighbour of x relative to X or a X - private neighbour of x. An X-private neighbour of x is either x itself, in which case x is an isolate vertex of $G\ [X]$, or is a neighbour of x in G which is not adjacent to any vertex of X . This latter type will be called an X-external private neighbour of x [3]. The set of all the external private neighbour of x relative to X is denoted by EPN (x, X) [3]. A set $D \subseteq V(G)$ is a super dominating set of G if EPN $(v, V(G) - D) \neq \emptyset$ for every vertex $v \in V(G) - D$ [3]. A super dominating set D of a graph G is said to be an SISD-set of G if $\langle D \rangle$ has at least one isolated vertex $a \in D$ such that $N_2(A) \cap D = \emptyset$, where $N_2(a) = \{ b : d(a, b) \leq 2$ and $a \neq b \}$. From the definition of strong isolate super domination, we initiated a new parameter "Unique Strong Isolate Super Domination(USISD)". The cardinality of a smallest unique strong isolate super dominating set of G, denoted by $\gamma_{0,sp}^{U,S}(G)$, is the unique strong isolate super domination number of G. In this paper, we establish several fundamental properties related to the unique strongly isolate super dominating set. In addition, we obtain the unique strong isolate super domination number for some families in graphs.

Definition 1.1: [5] A dominating set $D \subseteq V(G)$ in a graph $G = (V, E)$ is called a super dominating set if, for every $v \in V(G) - D$, there exists an external private neighbour of v with respect to $v \in V(G) - D$. A super dominating set D of a graph G is said to be an "Isolate Super Dominating Set"(ISD-set) of G has at least one isolated vertex.

Definition 1.2: [6] An ISD-set D is considered as Strong Isolate Super Dominating Set (SISD-set), if there exists a vertex $a \in D$ such that $N_2(a) \cap D = \emptyset$, where $N_2(a) = \{ b : d(a, b) \leq 2$ and $a \neq b \}$. The strong isolate super domination number(SISD-number), denoted by $\gamma_{0,sp}^{S}(G)$, is the minimum cardinality of a strongly isolate super dominating set of G.

Definition 1.3: An SISD-set D is considered as Unique Strong Isolate Super Dominating Set(USISD-set), if there exists exactly one strongly isolated vertex in D. The unique strong isolate super domination number(USISD-number), denoted by $\gamma_{0,sp}^{U,S}(G)$, is the minimum cardinality of a the unique strong isolate super dominating set of G.

2. Preliminary Results

In this section, we derive several fundamental results concerning the unique strong isolate super dominating sets in graphs. In particular, we characterize the graphs for which $\gamma_{0,sp}^{U,S}(G) = 1$.

Theorem 2.1: A graph G admits a USISD-set of size one if and only if $G \cong K_1$ or $G \cong K_2$.

Proof: Assume that G has USISD-set D with $|D| = 1$. Let $D = v$. Since D is a super dominating set, every vertex in $V(G) - D$ must be adjacent to v and v must be the unique private neighbour for each such vertex. Hence, $|V(G) - D| \leq 1$. If $|V(G) - D| = 0$, then $G \cong K_1$. If $|V(G) - D| = 1$, then $G \cong K_2$.
Converse part is trivial. ■

Theorem 2.2: Let G be a connected graph of order n. Then $\gamma_{0,sp}^{U,S}(G) = \frac{n}{2}$ if and only if G admits USISD-set such that $E(D, V(G) - D)$ is a perfect matching.

Proof: Let D be a minimum USISD-set of G and let $u_1 \in D$ be a strongly isolated vertex. Since D is a SD-set, $|D| = |V - D| = \frac{n}{2}$. Let $u_1 \in D$ be a PN of $v_1 \in V(G) - D$. By the strongly isolated, $N(v_1) \in D = \{u_1\}$. Applying the same argument to every vertex of $V(G) - D$, we obtain a one to one correspondence between D and $V(G) - D$. Hence $E(D, V(G) - D)$ forms a perfect matching. Conversely, assume D is a a minimum USISD-set and a perfect matching. Thus D is a perfect set and so $|D| \leq |V(G)-D|$. Since D is USISD-set, $|D| \geq |V(G)-D|$. Thus $|D| = |V(G)-D|$.

■

Next, we discuss USISD-set with Semi Total Dominating set(STD-set).

Definition 2.1: [11] *A* set S of vertices in a graph G is said to be a semi total dominating set of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S.

Lemma 2.1: Let G be a disconnected graph with $n \geq 2$ components $G_1, G_2, G_3, \ldots, G_n$. Suppose that exactly the first k components $G_1, G_2, G_3, \ldots, G_k$ admits USISD-set. Then $\gamma_{0,sp}^{U,S}(G)$ is given by $\min_{1\leq i\leq r}\{t_i\}$, where $t_i = \gamma_{0,sp}^{U,S}(G_i) + \sum_{j=1, j\neq i}^{n} \gamma_{t_2,sp}\big(G_j\big)$ for $1 \leq i \leq r$.

Proof: Let $t_1 = \min_{1\leq i\leq r}\{t_i\}$. Let D be a minimum USISD-set of G_1 and S_i be an STSD-set of G_i for each j with $2 \leq i \leq n$. Then $D \cup \big(U_{i=2}^{n} S_i\big)$ is a USISD-set of G with cardinality $\gamma_{0,sp}^{U,S}(G_1) + \sum_{i=2}^{n} \gamma_{t_2,sp}(G_i)$ and so $\gamma_{0,sp}^{U,S}(G) \leq \gamma_{0,sp}^{U,S}(G_1) + \sum_{i=2}^{n} \gamma_{t_2,sp}(G_i) = t_1$. Let D be a minimal USISD-set of G. Then D must intersect $V\big(G_i\big)$ for each $1 \leq i \leq n$. In addition, there exists an integer j such that $D \cap V\big(G_j\big)$ is a minimal USISD-set of G_j and $1 \leq j \leq r$. Likewise, for each $1 \leq i \leq n$, $i \neq j$, the set $D \cap V\big(G_i\big)$ is a minimal STSD-set of G_i. Thus, $|D| \geq \gamma_{0,sp}^{U,S}\big(G_j\big) + \sum_{i=1, i\neq j}^{n} \gamma_{t_2,sp}(G_i) \geq t_1$. Therefore, $\gamma_{0,sp}^{U,S}(G) = \min_{1\leq i\leq r}\{t_i\}$.

A maximum matching in a graph is a matching that contains the largest possible number of edges such that no two edges share a common vertex. The matching number $\alpha'(G)$ of G is the size of a maximum matching.

Theorem 2.3: Let G be a graph of order n admitting a USISD-set. Then $\gamma_{0,sp}^{U,S}(G) \geq n - \alpha'(G)$.

Proof: Let D be a minimum USISD-set of G. Then $|D| = \gamma_{0,sp}^{U,S}(G)$. Consequently, the set $M = \{uv \in E(G) : u \in D, v \in V(G) - D\}$, $|M| \leq \alpha'(G)$. By the definition of maximum matching, we have $|M| = |V(G) - D|$. Therefore, $|V(G) - D| \leq \alpha(G)$, $|D| \geq n - \alpha(G)$. Since $|D| = \gamma_{0,sp}^{U,S}(G)$, the result follows.

3. Unique Strong Isolate Super Domination Number for Some Families of Graphs

In this section, we compute the precise values of USISD-number for several standard classes of graphs including Paths, Star graphs, Comb graphs and Sun graphs. We begin by establishing a collection of basic results that will serve as useful tools in the subsequent analysis.

Corollary3.1: [3]For a path P_n with $n \geq 3$, $\gamma_{sp}(P_n) = \Big/\frac{n}{2}\Big/$.

In observation 2.2 [3], the authors present the following result for any connected graph G

$$1 \leq \gamma(G) \leq \frac{n}{2} \leq \gamma_{sp}(G) \leq n-1 \qquad (1)$$

Since every SISD-set is an ISD-set and every USISD-set is an SISD-set, From the above observation, the next result follows.

Theorem 3.1: Let G be a graph of order n which is not an empty graph. Then, $1 \leq \gamma(G) \leq \frac{n}{2} \leq \gamma_{sp}(G) \leq \gamma_{0,sp}(G) \leq \gamma_{0,sp}^{S}(G) \leq \gamma_{0,sp}^{U,S}(G) \leq n$.

Theorem 3.2: For a path P_n with ($n \geq 4$) is

$$\gamma^{S}_{0,sp}(P_n)=\begin{cases}\left\lceil\frac{n+1}{2}\right\rceil, & \text{if n} \equiv 0,1,3(\text{mod}4\,)\\ \left\lceil\frac{n}{2}\right\rceil, & \text{otherwise}\end{cases}$$

Proof: Let $V\,(P_n\,)=\{v_1, v_2, \ldots,v_n\,\}$. By Theorem 3.1 and Corollary 3.1, $\gamma^{U,S}_{0,sp}(P_n\,) \leq \gamma^{S}_{0,sp}(P_n\,) \geq \gamma_{\,0,sp}(P_n\,) \geq \gamma_{sp}\,(\,P_n\,)=\left\lceil\frac{n}{2}\right\rceil$. Note that $D = \{\,v_1\,\} \cup \{v_{\,4i}, v_{\,4i\,+\,1}\,/\,1 \leq i \leq k\,\}$ is a USISD-set with $\left\lceil\frac{n}{2}\right\rceil$ vertices when $n = 4k + 1$ or $4k + 2$, $k \geq 1, D = \{\,v_1\,,v_{n-1}, v_n\,\} \cup \{\,v_{4i}\,, v_{4i\,+\,1}\,/\,1 \leq i \leq k - 1\,\}$ is a USISD-set with $\left\lceil\frac{n+1}{2}\right\rceil$ vertices when $n = 4k$, $k \geq 1$ and $D = \{\,v_1, v_n\,\} \cup \{v_{4i}\,, v_{4i\,+\,1}/\,1 \leq i \leq k\,, k \geq 1\,\}$ is a USISD-set with $\left\lceil\frac{n}{2}\right\rceil$ vertices when $n = 4k + 3, k \leq 1$.

Theorem 3.3: Let G be a graph and D be a USISD-set of G. Suppose $e\,(v\,) \leq 2$ and v is not a full vertex of G, the $v \notin D$.

Proof: Suppose to the contrary, that $v \notin D$. Since $e\,(v\,) = 2$, every vertex of G is at distance at most 2 from v. As v is not a full vertex, there exists a vertex $u \in V\,(G\,)$ such that $d\,(u, v\,) = 2$. Hence there is a vertex w with $v - w - u$, a path of length 2. However, v does not dominate u and any vertex of D dominates u, which is contradiction to the unique strong isolated vertex. Thus u has no private neighbour in D, a contradiction. Hence, $v \notin D$.

Remark 3.1: Suppose D is an USISD-set of C_3. Let $V\,(C_3)=\{v_1, v_2,v_3\}$ and assume that v_1 is a strongly isolated vertex in D. There is no PN for either v_2 or v_3, a contradiction. Furthermore, Since $e\,(v\,)=2$ for all $v \in V\,(C_4)$, by Theorem 3.3, C_4 does not admit USISD-set.

Theorem 3.4: For any integer $n \geq 5$, the cycle C_n does not admit a USISD-set.

Proof: Suppose to the contrary, that C_n admits a USISD-set D. Let $V\,(C_n\,) = \{v_1,v_2,\ldots,v_n\,\}$ and assume that v_1 is a strongly isolated vertex in D. Since v_1 is strongly isolated vertex, none of its neighbours belong to D. Hence, $v_{n-1}, v_n\,,v_2, v_3 \notin D$. By the structure of cycle C_n and the strong isolation of v_1, the vertices v_n and v_2 cannot be adjacent to any vertex of D, a contradiction to the assumption that v_1 is the only strongly isolated vertex of D. Hence C_n does not admit a USISD-set for any $n \geq 5$.

We begin by examining the structure known as the comb graph, a graph that arises naturally in several graph-theoretic settings. This graph is constructed by starting with a path P_n and adjoining a single pendant vertex to each vertex of the path. As a result of this construction, the graph has 2 n vertices and exactly $2n - 1$ edges.

Lemma 3.1: [13] For any graph G of order $n,\gamma(G \odot K_1)=n$.

Theorem 3.5: Let $G = P_n \odot K_1$, be the comb graph of order $2n$, where $n \geq 3$. Then $\gamma^{U,S}_{0,sp}(G)=n$.

Proof: Let $V(G\,)=\{v_1,v_2,\ldots,v_n\,\} \cup \{\,u_1,u_2,\ldots,u_n\,\}$ and $E\,(G\,) = \{\,v_i\;v_{i+1}{:}1 \leq i\; \leq n{-}1\;\} \cup \{\,u_i\,v_i : 1 \leq i \leq n\,\}$. Here, $D = \{\,u_1,u_2,v_3,v_4,\ldots,v_n\,\}$ is a minimum USISD-set and u_1 is the only strongly isolated vertex in D. Since D is a minimum USISD-set, by Lemma 3.1 and Theorem 3.1, we get $\gamma^{U,S}_{0,sp}(G\,)=n$.

Next, we consider the Sun graph, which is formed by taking a cycle graph and appending a pendant edge to each of its vertices. The graph is commonly denoted by $S_n = C_n \odot K_1$.

Theorem 3.6: The n-sun graph $S_n = C_n \odot K_1$ does indeed admit a USISD-set with $\gamma^{U,S}_{0,sp}(S_n) = n$.

Proof: Let $V(S_n)=\{v_1,v_2,\ldots,v_n\} \cup \{u_1,u_2,\ldots,u_n\}$ and $E(G) = \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{u_i v_i : 1 \le i \le n\}$. Here $D = \{u_1,u_2,u_3,v_4,v_5,\ldots,v_n\}$ and u_2 is the only strongly isolated vertex in D. Since D is a minimum USISD-set, by Lemma 3.1 and Theorem 3.1, we get $\gamma^{U,S}_{0,sp}(S_n)=n$.

Theorem 3.7: If diam$(G) = 2$, then G admits USISD-set if and only if $G \cong P_1$ or $G \cong P_2$.

Proof: Suppose that G admits a USISD-set D. Let v be the only strongly isolated vertex in D. As D is a USISD-set, we have $d(u,v) \ge 3$. For all $u \in D$, since diam$(G) = 2$, this is possible only when $D = \{v\}$. Furthermore, for any USISD-set, $|D| \ge |V-D|$, which implies $|V(G)| \le 2$. Thus $G \cong P_1$ or $G \cong P_2$. The converse part is trivial.

Corollary 3.2: Let $n \ge 1$ be an integer. Then $K_{1,n}$ admits USISD-set iff $n = 1$.

Theorem 3.8: Let G be a connected graph of order $n \ge 2$ and let D be a USISD-set. Then the only strongly isolated vertex $v \in D$ has degree one.

Proof: Let v be the only strongly isolated vertex in D. If deg$(v) = 0$, then v will be an isolated vertex in a connected graph G of order $n \ge 2$, a contradiction. Suppose deg$(v) \ge 2$. Since v is a strongly isolated, $N(v) \cap D = \emptyset$. Thus there exist two distinct neighbours $x, y \in V - D$ such that x and y are adjacent with v. Thus v will not be a PN for both x and y. Thus there exist $u\ (\neq v) \in D$ such that u is a PN of x in D. Thus $d(u,v) = 2$ and $u \in D$, a contradiction to v is strongly isolated. Hence deg $(v)=1$.

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