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AN OPEN-TYPE GENERALIZED QUADRATURE RULE USING THE ANTI-GAUSSIAN APPROACH FOR NUMERICAL INTEGRATION

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Abstract: We propose an open-type Generalized Quadrature Rule by combining the Anti-Gauss 3-point rule with Simpson's 3/8 rule. The convergence properties of the new rule are rigorously analysed, and error estimates confirm its superior accuracy compared to the individual base rules. To validate these findings, we apply the rule to a range of test integrals. The results highlight the Generalized Quadrature Rule's improved performance and dominance over its constituents, particularly in handling indefinite integrals, making it a valuable tool for practical applications.

Keywords: Generalized quadrature rule, Anti-Gauss 3-point rule, Simpson 3/8 rule, $SM_{12}(f)$.
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1. Introduction

Numerical integration, a core aspect of numerical analysis, relies heavily on quadrature rules to approximate definite integrals effectively. The precision of these rules plays a pivotal role in determining their utility; higher precision often translates into more accurate results. Over the years, numerous mixed-type quadrature rules [7,8,9,10,11,14] have been developed, and ongoing research endeavours aim to enhance their precision further.

A notable advancement in this domain was introduced by S.K. Mohanty and R.B. Dash [5,6], who proposed a generalized methodology for achieving higher precision by combining multiple lower precision quadrature rules. Their work primarily focused on developing closed-type quadrature rules for evaluating definite integrals, providing a framework that has significantly influenced subsequent research in numerical integration.

This paper expands on the foundational work of Mohanty and Dash by further developing their generalized approach for open-type quadrature rules. In particular, we present a novel approach for deriving an open-type generalized quadrature rule with degree of precision five. This is achieved by amalgamating two existing lower precision rules: the anti-Gauss 3-point rule and Simpson's 3/8 rule [1,2,3,4,12,13,15], each with a precision of 3. The new rule serves as a powerful tool for enhancing accuracy in numerical integration, particularly in cases where open-type rules are preferred.

The remainder of this paper is organised in the following manner:

- **Section 2: Background and Motivation** – Explores the theoretical foundation and the need for higher precision quadrature rules.

- **Section 3: Review of Constituent Quadrature Rules** – Provides an overview of the anti-Gauss 3-point rule and Simpson's 3/8 rule, highlighting their precision and application.
- **Section 4: Development of the Generalized Quadrature Rule** – Details the methodology for constructing the precision-5 open-type quadrature rule by combining the constituent rules.
- **Section 5: Error Analysis** – Examines the accuracy and error characteristics of the newly developed rule.
- **Section 6: Applications and Case Studies** – Demonstrates the practical utility of the proposed rule through real-world numerical integration problems.
- **Section 7: Conclusions** – Summarizes the findings and discusses potential extensions to the proposed method.

Through this work, we aim to advance the field of numerical integration by introducing a novel, higher-precision open-type quadrature rule. This method addresses the challenges associated with achieving precision-5 in open-type rules and paves the way for further research and application in scientific computation.

2. Background and Motivation

A **generalized quadrature rule** is a higher-precision rule created by combining n quadrature rules, each with lower precision, where $n \in N$ and $n \geq 2$ [5,6].

Let SR_n represent this generalized quadrature rule of larger precision, constructed by merging the lower-precision quadrature rules $R_1, R_2, R_{3,} \ldots . R_n$, provided they satisfy the necessary SR-conditions [5,6]. The rule SR_n can then be expressed as follows:

$$SR_n = a_1R_1 + a_2R_2 + a_3R_3 + \cdots + a_nR_n;\ \sum_{i=1}^{n} a_i = 1 \tag{1}$$

Here, $a_1, a_2, a_3, \ldots a_n$ represent n rational coefficients, which are determined by ensuring that the rule SR_n is exact ∀ polynomials up to degree $dR_n + 2$. The truncation error associated with equation (1) is expressed as:

$$ESR_n = a_1ER_1 + a_2ER_2 + a_3ER_3 + \cdots + a_nER_n;\ \sum_{i=1}^{n} a_i = 1 \tag{2}$$

By assuming that the error vanishes for all polynomials up to degree SR_n, we can determine the coefficients $a_1, a_2, a_3, \ldots a_n$. Substituting these values into equation (1) allows us to derive the required generalized quadrature rule.

3. Review of Constituent Quadrature Rules

This section examines two established quadrature rules, both possessing a precision of 3.

3.1 Anti-Gaussian three-point rule

Using the approach proposed by Laurie [3], the anti-Gauss three-point quadrature is derived from the Gaussian 2-point rule. The Gauss two-point rule accurately integrates polynomials up to deg-3, while the anti-Gauss rule serves as its complement by handling polynomials that the Gaussian rule does not effectively address [3,4].

The anti-Gaussian rule aims to minimize errors in the integration of higher-degree polynomials, especially those orthogonal to the polynomials precisely handled by the Gauss rule. This leads to the formulation of the anti-Gaussian 3-point rule, which is expressed as:

$$aGL_3(f) = \frac{1}{13}\left[5f\left(-\sqrt{\frac{13}{5}}\right) + 16f(0) + 5f\left(\sqrt{\frac{13}{5}}\right)\right] \quad (3)$$

Applying Taylor's theorem to equation (3), we obtain:

$$aGL_3(f) = 2\left[f(0) + \frac{1}{3!}f''(0) + \frac{13}{9\times5!}f^{iv}(0) + \frac{169}{675\times6!}f^{vi}(0) + + \frac{(13)^3}{3\times8!\times(15)^3}f^{viii}(0) + \frac{2\times(13)^4}{10!\times(15)^5}f^{x}(0) + \cdots\right] \quad (4)$$

The exact value of the integral

Using Taylor's theorem [1,2,11,12,13,15], The integral's precise (exact) value is expressed as:

$$I(f) = 2\left[f(0) + \frac{1}{3!}f''(0) + \frac{1}{5!}f^{iv}(0) + \frac{1}{7!}f^{vi}(0) + + \frac{1}{9!}f^{viii}(0) + \frac{1}{11!}f^{x}(0) + \cdots\right] \quad (5)$$

Theorem 1

If $f(x)$ possesses sufficient differentiability on the interval $[-1,1]$, the truncation error corresponding to $aGL_3(f)$ can be expressed as follows:

$$EaGL_3(f) = -\frac{1}{135}f^{iv}(0) - \frac{1016}{7!\times675}f^{vi}(0) - \frac{6432}{9!\times(15)^3}f^{viii}(0) + \frac{131033}{11!\times(15)^5}f^{x}(0) + \cdots.$$

Proof: We have $I(f) = GL_3(f) + aGL_3(f)$ (6)

Using values from (4) and (5) on (6), we get

$$EaGL_3(f) = -\frac{1}{135}f^{iv}(0) - \frac{1016}{7!\times675}f^{vi}(0) - \frac{6432}{9!\times(15)^3}f^{viii}(0) + \frac{131033}{11!\times(15)^5}f^{x}(0) + \cdots. \quad (7)$$

Equation (7) confirms that the precision degree of $aGL_3(f)$ is three. □

3.1 Simpson's 3/8 Transformed rule

Simpson's 3/8 rule is a method [1,2,11,12] for numerical integration that approximates the value of a definite integral using cubic interpolation. For $\int_a^b f(x)dx$, the rule partitions the interval [a, b] into three subintervals of equal width, $h = \frac{b-a}{3}$, and the approximation is expressed as:

$$I(f) \approx \frac{3h}{8}[f(a) + 3f(a+h) + 3f(a+2h) + f(b)]$$

When the interval [a, b] is transformed to the standard interval $[-1,1]$, using a linear transformation $x = \frac{b-a}{2}t + \frac{b+a}{2}$, the Simpson's 3/8 Transformed rule becomes:

$$I(f) \approx SP_{\frac{3}{8}}(f) = \frac{1}{4}\left[f(-1) + 3f\left(-\frac{1}{3}\right) + 3f\left(\frac{1}{3}\right) + f(1)\right]$$

Due to Taylor's Theorem

$$SP_{\frac{3}{8}}(f) = 2f(0) + \frac{1}{3}f''(0) + \frac{7}{324}f^{iv}(0) + \frac{61}{4\times21870}f^{vi}(0) + \frac{3281}{2^7\times103335}f^{viii}(0) + \cdots \quad (8)$$

The truncation error arises from neglecting higher-order terms of the Taylor expansion. For Simpson's 3/8 rule, the truncation error $ESP_{\frac{3}{8}}(f)$ is provided in Theorem 2.

Theorem 2

If $f(x)$ possesses sufficient differentiability on the interval $[-1,1]$, the truncation error corresponding to $SP_{\frac{3}{8}}(f)$ can be expressed as follows:

$$ESP_{\frac{3}{8}}(f) = -\frac{2}{405} f^{iv}(0) - \frac{23}{35\times(3)^7} f^{vi}(0) - \frac{60633}{9!\times 6889} f^{viii}(0) + \cdots.$$

Proof: We have $I(f) = SP_{\frac{3}{8}}(f) + ESP_{\frac{3}{8}}(f)$ (9)

Using values from (8) and (5) on (9), we get

$$ESP_{\frac{3}{8}}(f) = -\frac{2}{405} f^{iv}(0) - \frac{23}{35\times(3)^7} f^{vi}(0) - \frac{60633}{9!\times 6889} f^{viii}(0) + \cdots. \quad (10)$$

The error term in equation (10), expressed in rational coefficient form, shows that the rule accurately integrates polynomials up to degree 3 but introduces errors for higher-degree terms. Therefore, the precision degree of Simpson's 3/8 rule is 3. □

4. Construction of Proposed $SM_{12}(f)$ Generalized quadrature rule

This section outlines the construction of a generalized quadrature rule [5,6] using the methodology described in (1) and (2). It integrates two lower-precision rules through a step-by-step combination technique, ensuring clarity in formulation. A generalized quadrature rule of order 2 is derived by blending these two rules using the generalized quadrature framework. Here we consider,

$R_1(f) = aGL_3(f)$ = Anti-Gaussian 3-point quadrature rule.

$R_3(f) = SP_{3/8}(f)$ = Simpson's 3/8 rule Transformed quadrature rule.

Since each rule has a precision of 3, it follows that $dR_1(f) = dR_2(f)$. Consequently, $R_1(f)$ and $R_2(f)$ satisfy the SR conditions [5,6]. The formulation of the proposed generalised quadrature rule is presented in Theorem 3.

Theorem 3

If $f(x)$ possesses sufficient differentiability on the interval $[-1,1]$, the generalized quadrature $SM_{12}(f)$ can be expressed as:

$$SM_{12}(f) = 3SP_{3/8}(f) - 2aGL_3(f)$$

The corresponding truncation error is given by:

$$ESM_{12}(f) = 3ESP_{3/8}(f) - 2EaGL_3(f).$$

Proof: The generalized quadrature rule $SM_{12}(f)$, expressed in terms of $R_1(f)$ and $R_2(f)$, is given by:

$$SM_{12}(f) = [a_1 R_1(f) + a_2 R_2(f)], \text{ where, } a_1 + a_2 = 1 \quad (11)$$

The corresponding truncation error for $SM_{12}(f)$ is given by:

$$ESM_{12}(f) = [a_1 ER_1 + a_2 ER_2] \quad (12)$$

By applying equations (7) and (11) to equation (12), we obtain:

$$ESM_{12}(f) = a_1\left\{-\frac{1}{135} f^{iv}(0) - \frac{1016}{7!\times 675} f^{vi}(0) - \frac{6432}{9!\times(15)^3} f^{viii}(0) + \frac{131033}{11!\times(15)^5} f^{x}(0) + \cdots\right\} + a_2\left\{-\frac{2}{405} f^{iv}(0) - \frac{23}{35\times(3)^7} f^{vi}(0) - \frac{60633}{9!\times 6889} f^{viii}(0) + \cdots\right\} \quad (13)$$

We select a_1 and a_2 such that the quadrature rule $SM_{12}(f)$ is exact for all polynomials up to degree 5, i.e., dR_2+2=5.

From the error term, we derive the equation:

$$\frac{a_1}{135} + \frac{2a_2}{405} = 0 \tag{14}$$

Solving equations (11) and (14) yields $a_1 = -2$ and $a_2 = 3$.
Substituting these values into equations (11) and (12) leads to the desired result.

Corollary: If $f(x)$ possesses sufficient differentiability on the interval $[-1,1]$, the truncation error corresponding to $SM_{12}(f)$ can be expressed as follows:

$$ESM_{12}(f) = -\frac{3131}{7!\times 5^4 \times 3^4} f^{vi}(0) - \frac{175096343}{9!\times 15^2 \times 20667} f^{viii}(0) - \cdots$$

Proof: Substituting the values of a_1 and a_2 into equation (13) yields the desired result. □

Pictorial Representation of formulation of the rule

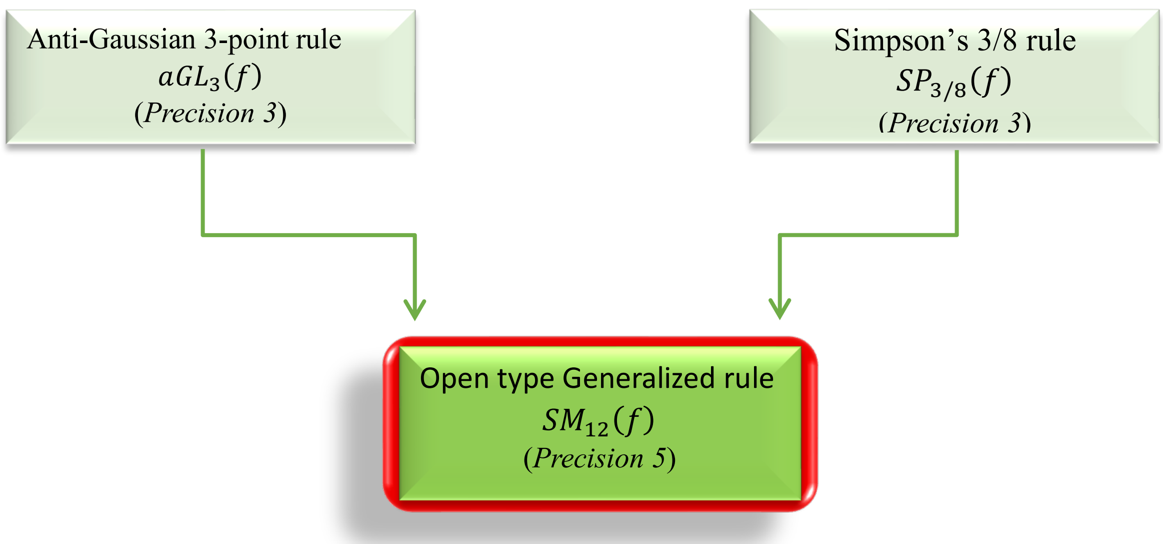

Figure-1: *Formulation of* $SM_{12}(f)$ *the rule*

5. Error Analysis

Following from Theorem 1 and the corollary of Theorem 3

$$|ESM_{12}(f)| \leq |EaGL_3(f)|$$

Following from Theorem 2 and the corollary of Theorem 3

$$|ESM_{12}(f)| \leq |ESP_{\frac{3}{8}}(f)|$$

A comparison between the proposed precision-5 rule and existing lower-precision rules highlights the improved accuracy of our generalized method. Both theoretical justification and practical examples supporting this enhancement are provided in Section 6.

Theorem 4

The error bound of the constructed quadrature rule $SM_{12}(f)$ is

$$\Rightarrow |ESM_{12}(f)| \leq \frac{2M}{135} |\xi_2 - \xi_1|,\ \xi_1, \xi_2 \epsilon [-1, 1] \text{ and } M = \sup_{-1\leq x\leq 1} |f^{v}(x)|$$

Proof: From (7), we get $EaGL_3(f) \cong -\frac{1}{135} f^{iv}(\xi_1), \quad \xi_1 \epsilon [-1, 1]$

and from (10), we get $ESP_{\frac{3}{8}}(f) \cong -\frac{2}{405} f^{iv}(\xi_2), \quad \xi_2 \epsilon [-1, 1]$

using above vales on (12) and using values of $a_1$and a_1, we get

$$ESM_{12}(f) = \frac{2}{135} f^{iv}(\xi_1) - \frac{6}{405} f^{iv}(\xi_2) = \frac{2}{135} \{f^{iv}(\xi_1) - f^{iv}(\xi_2)\}$$

Due to Lagrange's Mean value theorem

$ESM_{12}(f) \cong \frac{-2}{135}(\xi_2 - \xi_1)\, f^{v}(\xi)$, for some $\xi \, \epsilon \, (\xi_1, \xi_2)$, when $\xi_1 \le \xi_2$ (otherwise reverse the order)

$\Rightarrow |ESM_{12}(f)| \cong \frac{2}{135} |(\xi_2 - \xi_1)\, f^{v}(\xi)| \le \frac{2}{135} |\xi_2 - \xi_1| |\, f^{v}(\xi)|$

$$\le \frac{2}{135} |\xi_2 - \xi_1|\, K, \text{where } K = \sup_{\xi_1 \le x \le \xi_2} |f^{v}(x)|$$

$$\le \frac{2}{135} |\xi_2 - \xi_1|\, M, where\ M = \sup_{-1 \le x \le 1} |f^{xi}(x)| \text{ and } K \le M$$

$$\Rightarrow |ESM_{12}(f)| \le \frac{2M}{135} |\xi_2 - \xi_1| \qquad (15)$$

Since ξ_1 *and* ξ_2 are arbitrarily chosen points in $[-1, 1]$, (15) shows that the absolute value of the truncation error will be less if the points ξ_1 *and* ξ_2 are closure to each other. ☐

Corollary.

The error bound for the truncation error is

$|ESM_{12}(f)| \le \frac{4M}{135}, M = \sup_{-1 \le x \le 1} |f^{v}(x)|$

Proof: From the theorem 4

$$|ESM_{12}(f)| \le \frac{2M}{135} |\xi_2 - \xi_1|, \ \ \xi_1, \xi_2 \epsilon [-1, 1], \text{ where } M = \sup_{-1 \le x \le 1} |f^{v}(x)|$$

Again $|\xi_2 - \xi_1| \le 2$, ref [1]. Using on the above inequation, we have

$$|ESM_{GLKEL}(f)| \le \frac{4M}{135}$$ ☐

6. Applications and Case Studies

This section examines the new quadrature rule's application to open-type test integrals, analysing five examples. Results and error comparisons are presented in Tables 1 and 2, respectively.

Table 1: Values computed from 5 test integrals.

Integral	$aGL_3(f)$	$SP_{\frac{3}{8}}(f)$	$SM_{12}(f)$
$I_1 = \int_0^\infty e^{-x}\sqrt{x}\, dx$	0.886227119	0.886227079015	0.886226999045
$I_2 = \int_0^\infty \frac{\sin x}{x e^x} dx$	0.78539947	0.785398998997	0.785398056991
$I_3 = \int_0^\infty \frac{\log(1+x)}{e^x} dx$	0.596347895	0.5963478310979	0.5963477032937

$I_4 = \int_0^\infty \frac{\log x}{e^x} dx$	−0.577215756	−0.57721572990 1	−0.577215677703
$I_5 = \int_0^\infty \frac{dx}{e^x(1+x^2)}$	0.621449289	0.60912144977	0.58446577131

Table 2: Error comparison

Integral	Exact Value	$\lvert EaGL_3(f)\rvert$	$\lvert ESP_{3/8}(f)\rvert$	$\lvert ESM_{12}(f)\rvert$
I_1	0.886226925452758	1.9354724×10^{-7}	1.5356×10^{-7}	7.3592×10^{-8}
I_2	0.785398163397030	1.3066×10^{-6}	8.356×10^{-7}	1.064×10^{-8}
I_3	0.596347362323194	5.326768×10^{-7}	4.6877×10^{-7}	3.4097×10^{-7}
I_4	−0.577215664901532	9.1098468×10^{-8}	6.4999468×10^{-8}	1.28014×10^{-8}
I_5	0.577158665928169	4.4290623×10^{-2}	3.196278×10^{-2}	7.307105×10^{-3}

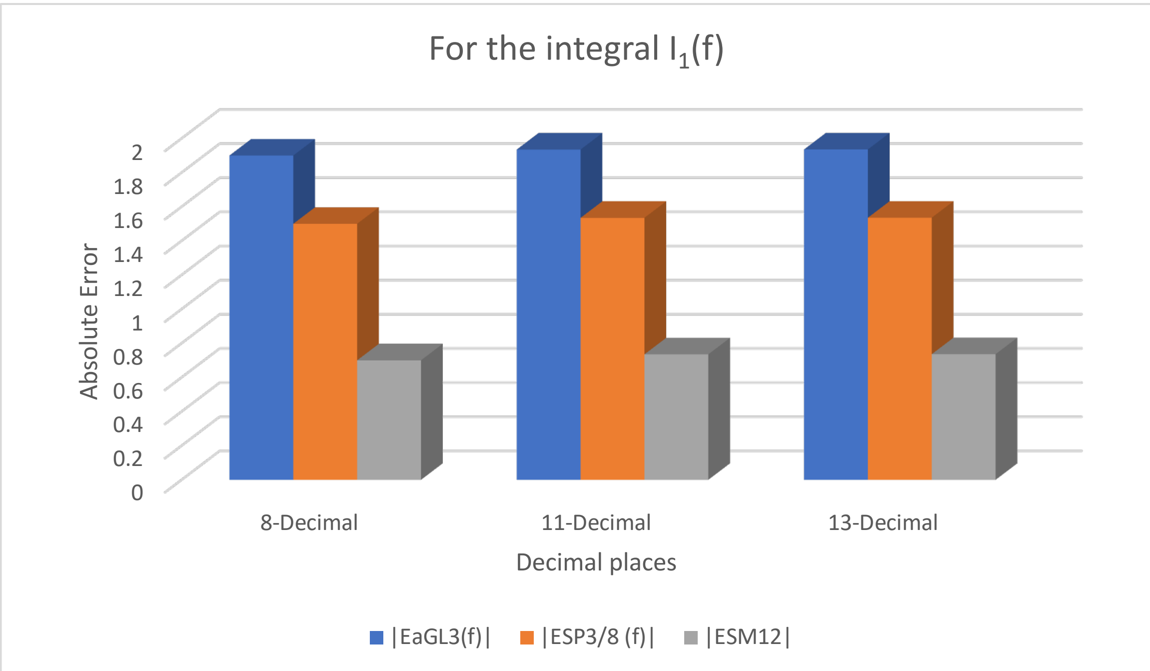

Figure 2: Error comparison for $I_1(f)$

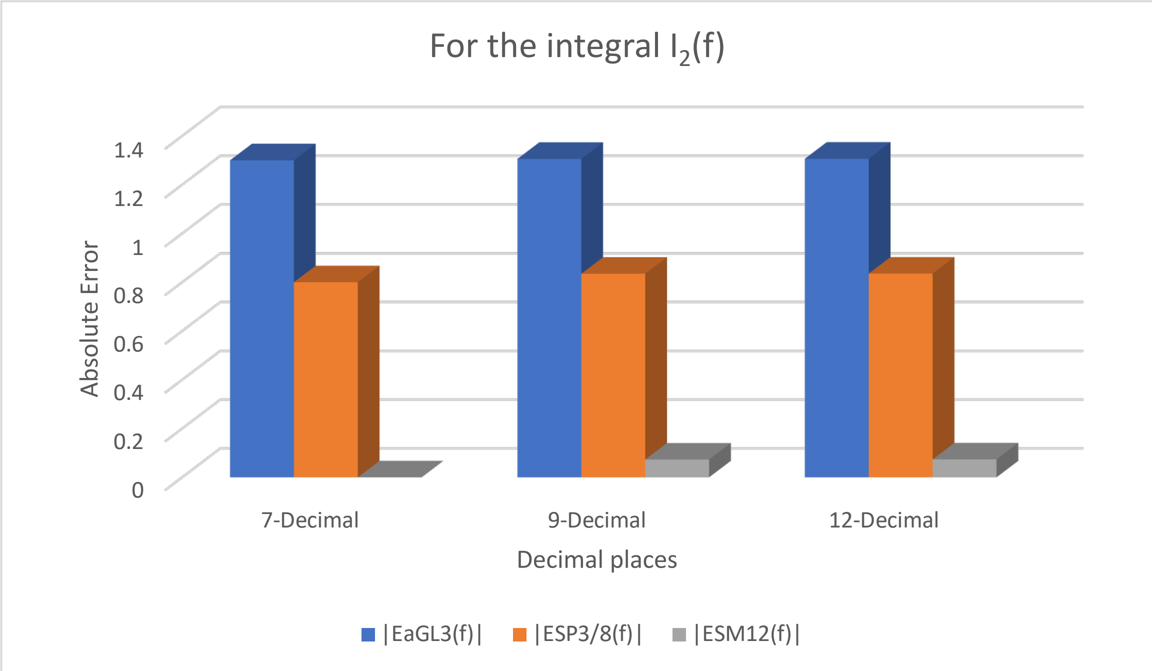

Figure 3: Error comparison for $I_2(f)$

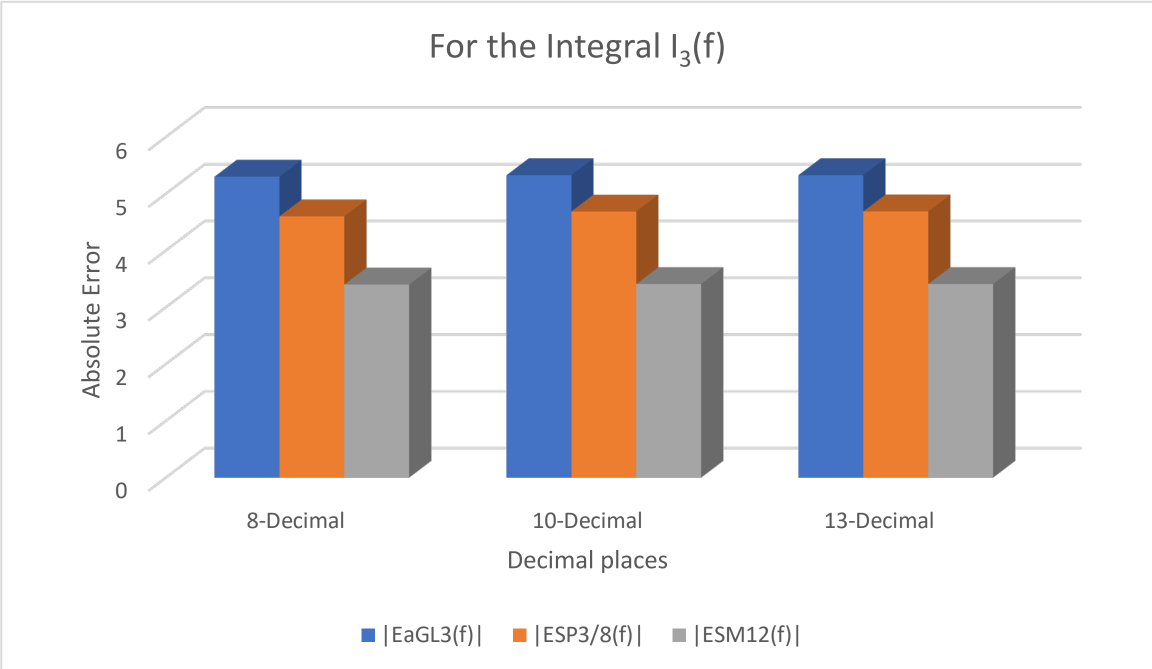

Figure 4: Error comparison for $I_3(f)$

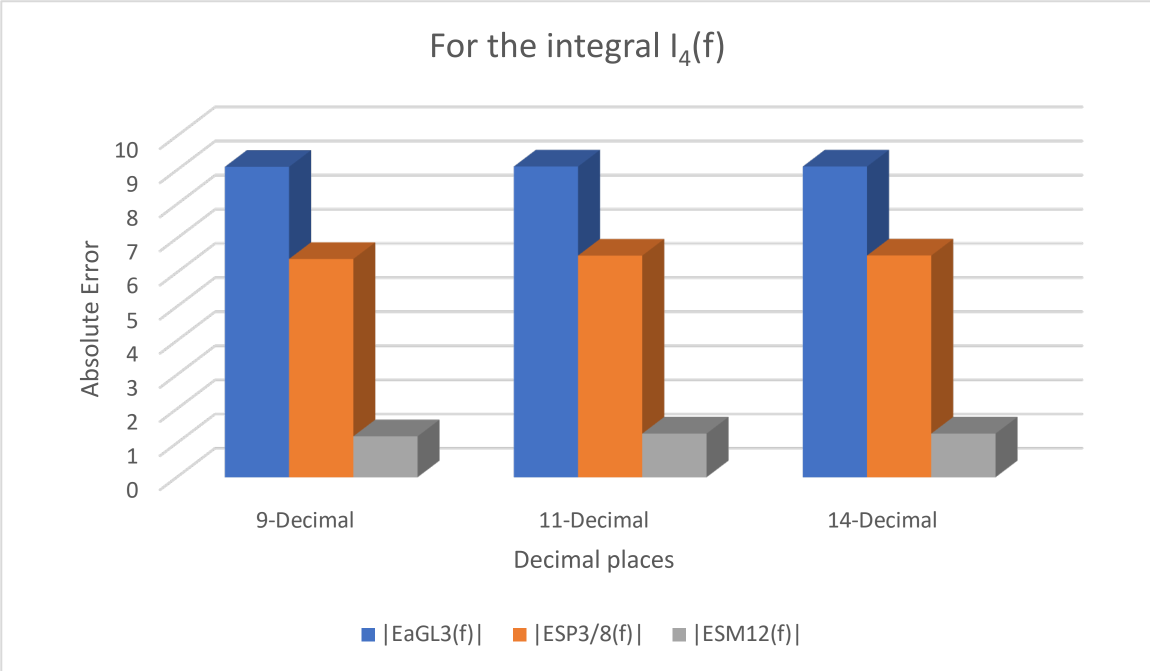

Figure 5: Error comparison for $I_4(f)$

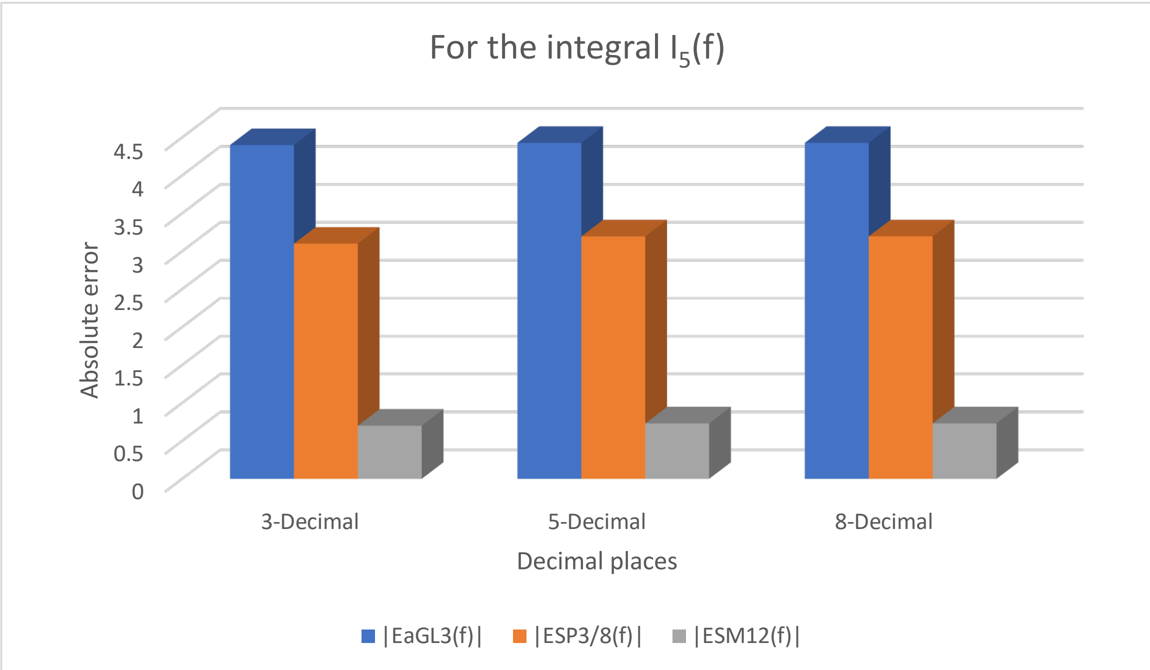

Figure 6: Error comparison for $I_5(f)$

Observation: The above five figures are drawn based on the data available in the Table 2. We observe that the error due to constructed $SM_{12}(f)$ is less in comparison to its constituents.

6. Conclusions.

The constructed rule significantly outperforms base rules in numerical integration, reducing errors both theoretically and practically. The introduction of an open-type generalized quadrature rule with precision-5 represents a significant advancement in the field. By integrating the Anti-Gaussian 3-point rule with Simpson's 3/8 rule, this approach enhances accuracy and efficiency, providing a practical solution for high-precision integration. This contribution holds promise for improving a variety of numerical analysis applications through enhanced precision and reliability.

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